Asymptotics of characters and large Young diagrams

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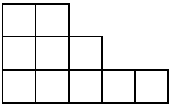
Context

Question

Asymptotic behavior of some models of random Young diagrams ?

In other terms:

- For each n, we give a probability measure on Young diagrams of size n.



A Young diagram of size 10.

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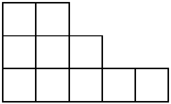

- Questions: convergence of the first k rows? Convergence of some other parameters? Continuous limit shape after some rescaling?

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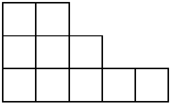

- Questions: convergence of the first k rows? Convergence of some other parameters? Continuous limit shape after some rescaling?

For some measures, representation theory of symmetric groups is a good tool to answer these questions.

Outline of the talk

1. Representations of symmetric groups and functions on Young diagrams.

2. q-Plancherel measure
   - Plancherel measure
   - A q-deformation
   - Asymptotic behavior

Irreducible character values

Fact

Irreducible representations (V, ρ) of the symmetric group are (canonically) indexed by Young diagrams λ of size n.

If $\sigma \in S_n$, we look at its *normalized trace* on V_λ, *i.e.* :

$$\chi^\lambda(\sigma) = \frac{\mathrm{Tr}\left(\rho_\lambda(\sigma)\right)}{\dim(V_\lambda)}.$$

Polynomial functions on the set of Young diagrams

Let $\sigma \in \mathfrak{S}_k$. We define the following function on **all** Young diagrams:

$$\mathrm{Ch}_\sigma(\lambda) = \begin{cases} n^{\downarrow k} \chi^\lambda(\tilde{\sigma}) & \text{if } \lambda \vdash n \geq k \\ 0 & \text{if } \lambda \vdash n < k \end{cases}$$

where $n^{\downarrow k} = n(n-1)\dots(n-k+1)$
and $\tilde{\sigma}$ is the image of σ by the canonical inclusion $S_k \hookrightarrow S_n$ (we just add fixed points to have a permutation in S_n).

Theorem

The random variables Ch_σ span linearly a $\mathbb{C}$-algebra $\mathcal{O}$.

Example: $\mathrm{Ch}_{(2)} \cdot \mathrm{Ch}_{(2)} = 4 \cdot \mathrm{Ch}_{(3)} + \mathrm{Ch}_{(2,2)} + 2\mathrm{Ch}_{(1,1)}$.

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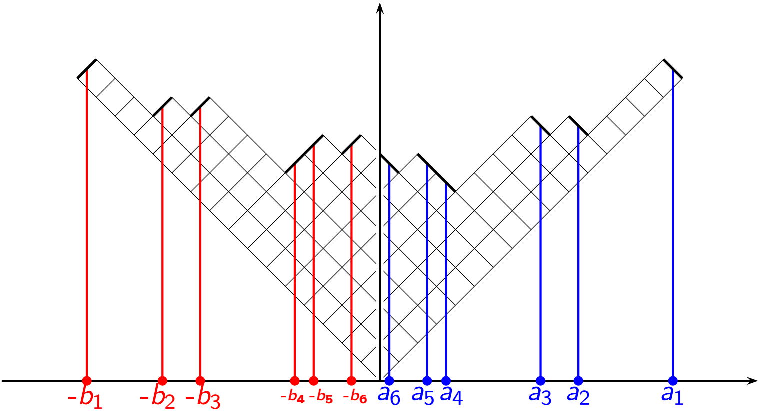
We will describe an algebraic basis of this algebra.

Frobenius coordinates and their power sums

If λ is a Young diagram, define its Forbenius coordinates $(a_i, b_i), 1 \leq i \leq h$ as follows:



$$a_i = \lambda_i - i + 1/2 > 0, \quad b_i = \lambda'_i - i + 1/2 > 0$$

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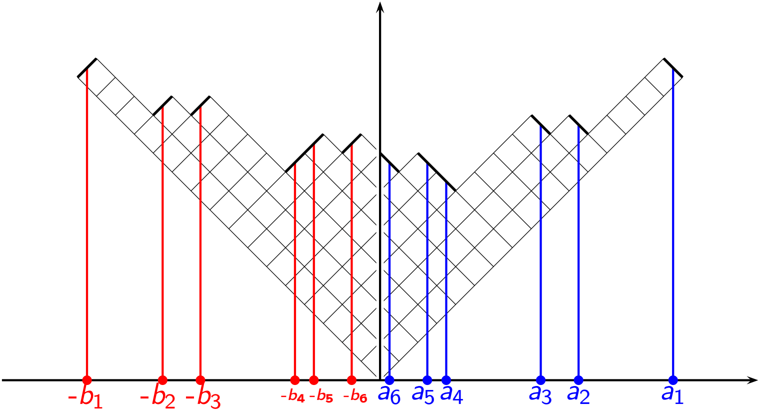

$$p_m(\mathbb{F}_\lambda) := \sum a_i^m - (-b_i)^m$$

Character values in terms of Frobenius coordinates

Fix a permutation $\sigma \in S_k$.
Denote $\ell_1, \dots, \ell_r$ the length of its cycles.

Theorem

For all Young diagrams λ,

$$Ch_\sigma(\lambda) = \prod_{j=1}^{r} p_{\ell_j}(\mathbb{F}_\lambda) + P_\sigma(p_1(\mathbb{F}_\lambda), p_2(\mathbb{F}_\lambda), \dots),$$

where P_σ is a polynomial in variables $p_1, p_2, \dots$ of degree smaller than k (by definition, $\deg(p_m) = m$) which does not depend on λ.

Ex: $\mathrm{Ch}_{(1\ 2\ 3\ 4)} = p_4 - 4p_2 \cdot p_1 + 11/2\ p_2$.
$\mathrm{Ch}_{(1\ 2\ 3)(4\ 5)} = p_3 \cdot p_2 + 6p_4 - 3/2\ p_2 \cdot p_1^2 - 67/4\ p_2 p_1 + 21 p_2$.

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where P_σ is a polynomial in variables $p_1, p_2, \dots$ of degree smaller than k (by definition, $\deg(p_m) = m$) which does not depend on λ.

- Explicit formula in the one-cycle case (Wasserman, 1981);
- Easy to extend to the general case using Faharat-Higman algebra (1957).

Inverting the previous formula

Consequence: if we define

$$\mathcal{O}_d = \mathrm{Vect}\left(\bigcup_{1 \leq k \leq d} \bigcup_{\sigma \in S_k} \mathrm{Ch}_\sigma \right),$$

then $\mathcal{O} = \bigcup \mathcal{O}_d$ is a filtered algebra.

$p_m(\mathbb{F}_\lambda)$ can be expressed as a linear combination of Ch_σ. Ex:

$$p_3 = \mathrm{Ch}_{(1\ 2\ 3)} + 3/2\ \mathrm{Ch}_{(1)(2)} + 1/4\ Ch_{(1)};$$
$$p_2^2 = \mathrm{Ch}_{(1\ 2)(3\ 4)} + 4\mathrm{Ch}_{(1\ 2\ 3)} + 2\mathrm{Ch}_{(1)(2)}.$$

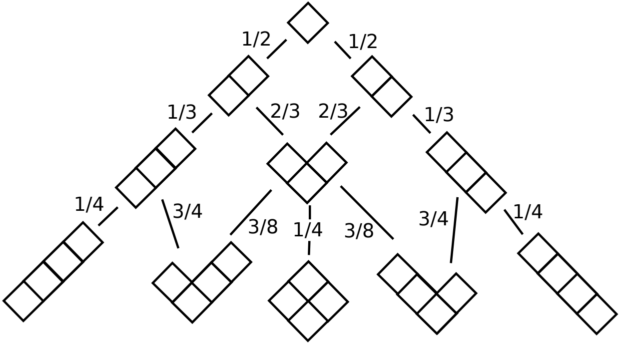
$$\left(\lambda \mapsto \prod p_{m_i}(\mathbb{F}_\lambda)\right) = \mathrm{Ch}_\sigma + \text{smaller degree terms},$$

where σ is a permutation with cycles of length $m_1, \dots, m_r$.

The Plancherel measure

$\mathcal{P}_n$: a measure on Young diagrams of size n.

1. can be defined by a Markov process:



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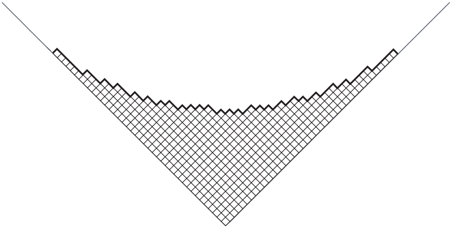
Example of large random Young diagram:



limit shape: Kerov and Vershik / Logan and Shepp (1977);
Fluctuations: Kerov(1993), Ivanov-Olshanski (2003).

The Plancherel measure

$\mathcal{P}_n$: a measure on Young diagrams of size n.

2. can be defined, using representation theory:

- $\mathcal{P}_n(\{\lambda\}) = \frac{(\dim V_\lambda)^2}{n!}$.
- Recall:

$$\mathbb{C}[\mathfrak{S}_n] \simeq \bigoplus_{\lambda \vdash n} V_\lambda^{\dim V_\lambda}$$

(action on $\mathbb{C}[\mathfrak{S}_n]$: $\tau \cdot (\sum_\sigma c_\sigma \sigma) = \sum_\sigma c_\sigma \tau \circ \sigma$.)

- In this context :

$$\mathcal{P}_n(\{\lambda\}) = \frac{\dim(\text{isotypic component of type } \lambda)}{\dim \mathbb{C}[\mathfrak{S}_n]}$$

Normalized character values have simple expectations!

Let $\sigma \in \mathfrak{S}_k$. If $n \geq k$ and $\lambda \vdash n$, recall that

$$\mathrm{Ch}_\sigma(\lambda) = n(n-1)\dots(n-k+1) \cdot \chi^\lambda(\tilde{\sigma})$$

Ch_σ can be seen as a *random variable*. Let us compute its expectation:

$$\begin{aligned}\mathbb{E}_{\mathcal{P}_n}(\mathrm{Ch}_\sigma) &= \frac{n^{\downarrow k}}{n!} \sum_{\lambda \vdash n} (\dim V_\lambda) \cdot \mathrm{Tr}_{V_\lambda}(\tilde{\sigma}) \\ &= \frac{n^{\downarrow k}}{n!} \mathrm{Tr}_{\left(\bigoplus_{\lambda \vdash n} V_\lambda^{\dim V_\lambda}\right)}(\tilde{\sigma}) = \frac{n^{\downarrow k}}{n!} \mathrm{Tr}_{\mathbb{C}[\mathfrak{S}_n]}(\tilde{\sigma}) = n^{\downarrow k} \, \mathrm{tr}_{\mathbb{C}[\mathfrak{S}_n]}(\tilde{\sigma})\end{aligned}$$

Last expression is easy to evaluate:

$$\mathbb{E}_{\mathcal{P}_n}(\mathrm{Ch}_\sigma) = n^{\downarrow k} \, \delta_{\sigma, \mathrm{Id}_k}$$

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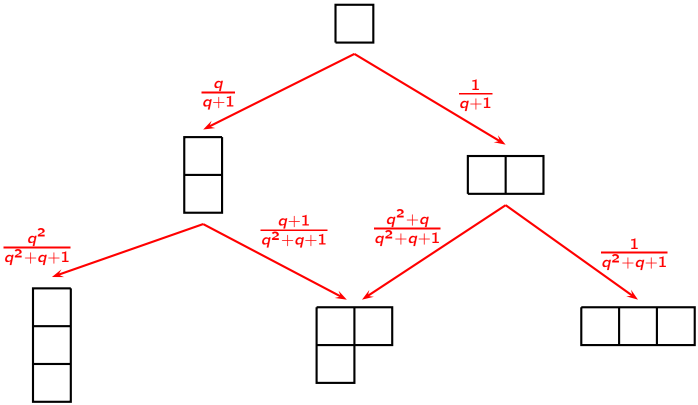

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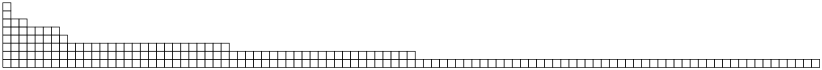
1. example of a large random diagram



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2. can be defined, using representation theory of Hecke algebras:

- Similarly to Plancherel measure, one has

$$\mathbb{E}_{q\text{-}\mathcal{P}_n}\left(\chi^{q,\bullet}(T_\sigma)\right) = 0 \quad \text{if } \sigma \neq 0,$$

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- One can translate this with usual characters:

$$\mathbb{E}_{q\text{-}\mathcal{P}_n}\left(Ch_\sigma\right) = \frac{(1-q)^k}{\prod_j (1-q^{\ell_j})} n^{\downarrow k}.$$

Filtration and order of magnitude

Lemma

$$x \in \mathcal{O}_d \;\Rightarrow\; \mathbb{E}(x) = O(n^d)$$

Proof: true on the generating family Ch_σ.

Application:

$$\mathbb{E}_{q\text{-}\mathcal{P}_n}(p_m(\mathbb{F}_\lambda)) \sim \mathbb{E}_{q\text{-}\mathcal{P}_n}\left(Ch_{(1\ \dots\ m)}\right) \sim \frac{(1-q)^m}{1-q^m} n^m$$

$$\mathrm{Var}_{q\text{-}\mathcal{P}_n}(p_m(\mathbb{F}_\lambda)) = O(n^{2m-1})$$

$\frac{p_m(\mathbb{F}_\lambda)}{n^m}$ converges in probability towards $\frac{(1-q)^m}{1-q^m}$.

Convergence of the first rows

But $\frac{p_m(\mathbb{F}_\lambda)}{n^m}$ is the $(m-1)$-th moment of the probability measure

$$X_\lambda = \sum_{i=1}^{d} (a_i^*(\lambda)/n)\, \delta_{(a_i^*(\lambda)/n)} + (b_i^*(\lambda)/n)\, \delta_{(-b_i^*(\lambda)/n)}.$$

and $\frac{(1-q)^m}{1-q^m}$ the $(m-1)$-th moment of

$$X_\lambda = \sum_{i \geq 1} q^{i-1}(1-q)\delta_{q^{i-1}(1-q)}$$

We have convergence in probability of the repartition function at each point $x \neq q^{i-1}(1-q)$

Theorem (F., Méliot 2010)

For every $i \geq 1$, in probability

$$\lambda_i/n \longrightarrow_{q\text{-}\mathcal{P}} q^{i-1}(1-q)$$

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- Quite simple method:
  1. Compute expectation of character value;
  2. Deduce the convergence of some parameters (easy!);
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  3. Translate it on the shape of the Young diagram.
- This result could be deduced directly from step 1 using Martin boundary theory.
- One can also obtain second-order asymptotics.
- Ideas come from a method of Kerov-Ivanov-Olshanski to study fluctuations of Young diagrams under Plancherel measure

End of the talk

Thanks for listening.

Questions?