Linear coefficients of Kerov's polynomials: bijective proof and refinement of Zagier's result

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Introduction

Question

Let $m \leq N$ with $m \equiv N \pmod 2$. What is the number $B(N,m)$ of permutations σ of size N:

- with m cycles (notation : $\kappa(\sigma) = m$) ;
- such that $(1\ 2\ \dots N)\sigma^{-1}$ is a long cycle?

Motivations

- particular cases of coefficients of some character polynomials (ask for details!).
- surprising formula (Zagier [4], 1995) :

$$\frac{N(N+1)}{2}B(N,m) = |\{\sigma \in S_{N+1} \text{ avec } \kappa(\sigma) = m\}| \qquad (1)$$
$$=: A(N+1,m) \quad \text{(Stirling number)}$$

Main result

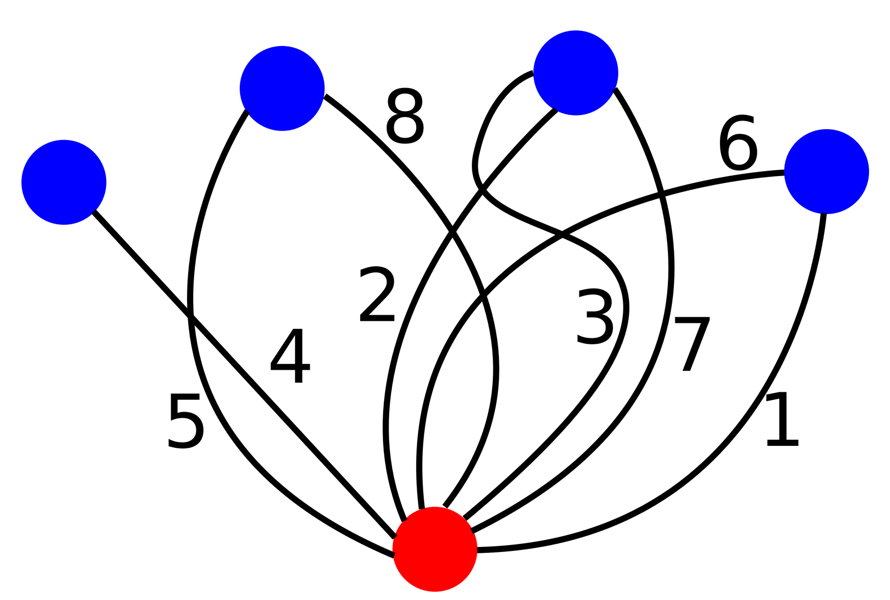
- A combinatorial proof of (1) (asked by Stanley [3], 2009).
- A refinement taking the cycle type of permutations into account.

Reformulation using maps

Permutations $\simeq$ Unicellular red-rooted bipartite maps

$\sigma = (1\ 6)(2\ 7\ 3)(4)(5\ 8), \quad (1\ 2\ \dots N)\sigma^{-1} = (1\ 7\ 3\ 8\ 6\ 2\ 4\ 5)$

$\leftrightarrow$



$$\Rightarrow B(N,m) = \left| \left\{ \begin{array}{c} \text{rooted unicellular bipartite maps} \\ \text{with } N \text{ edges} \\ \text{with 1 red and } m \text{ blue vertices} \end{array} \right\} \right|$$

$$A(N+1,m) = \left| \left\{ \begin{array}{c} \text{rooted unicellular bipartite maps} \\ \text{with } N+1 \text{ edges} \\ \text{and } m \text{ blue vertices} \end{array} \right\} \right|$$

Is there a bijection explaining equation (1)?

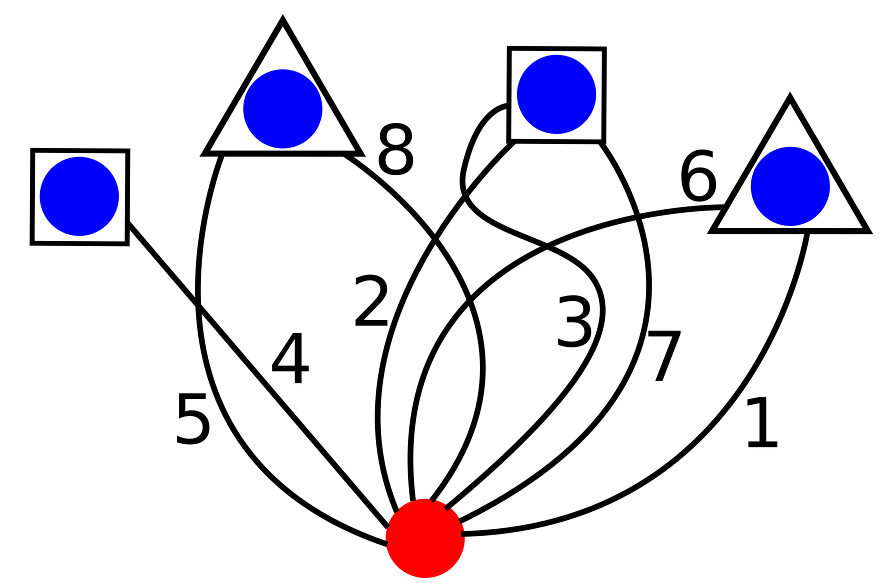
☹ We didn't manage to construct a direct one.

Blue-partitioned maps and star thorn trees

Equivalent statement

Definition 1. *A blue-partitioned map is a map with a partition of its blue vertices.*

- $C(N,m) := \#$ *rooted unicellular bipartite map with N edges and m blocks of blue vertices.*
- $D(N,m) := \#$ *same objects with only one red vertex.*



Proposition 2.

$$\sum_p C(N,p)(x)_p = \sum A(N,m)x^m$$
$$\sum_p D(N,p)(x)_p = \sum B(N,m)x^m$$

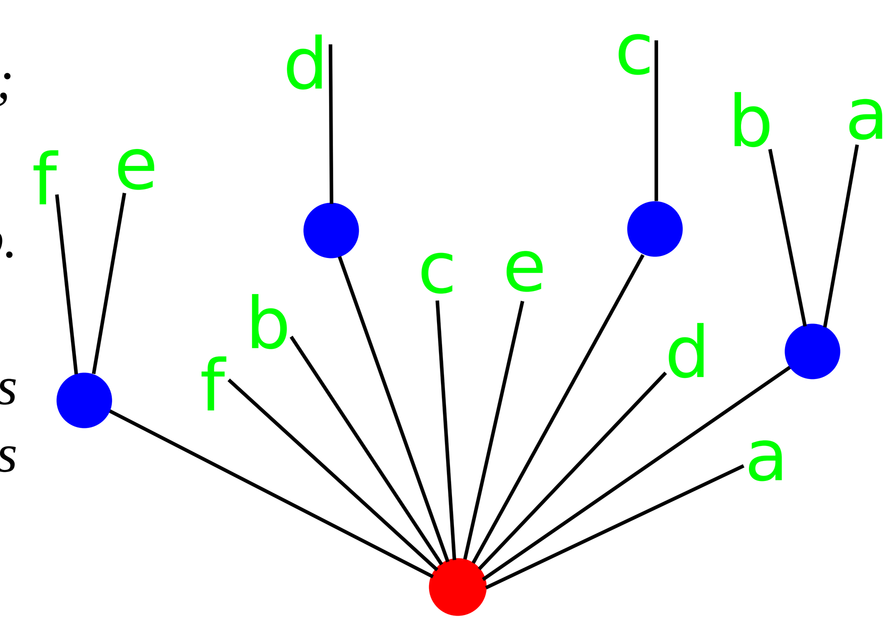
Remark 3. *The refined version uses symmetric functions!*

$\Rightarrow$ Equation (1) is equivalent to:

$$\forall\, p \leq N,\ N(N+1)D(N,p) = C(N+1,p) \qquad (2)$$

Theorem 4 (Morales, V. [2], 2009)**.** *The quantity $C(N,p)$ is also the number of permuted star thorn trees i.e. bicolored trees with:*

- *only 1 red vertex (the root);*
- *p blue vertices;*
- *$N-p$ thorns of blue (resp. red) extremity;*
- *a bijection between thorns of blue extremity and thorns of red extremity.*



Corollary 5.

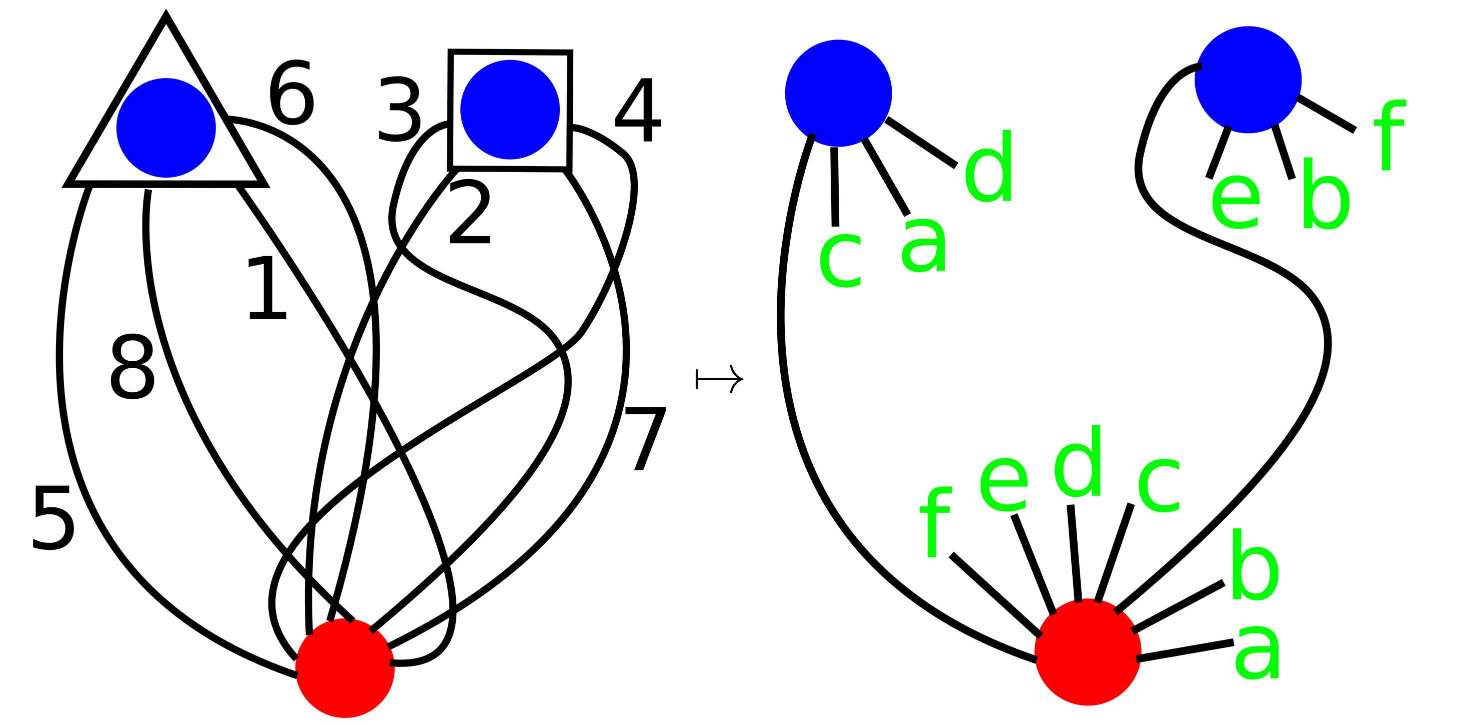
$$(N+1-p)C(N+1,p) = N(N+1)C(N,p)$$

Therefore, Equation (1) $\Leftrightarrow$ $(N+1-p)D(N,p) = C(N,p)$

Combinatorial construction

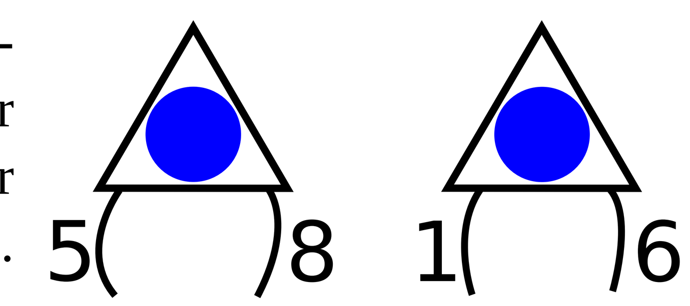
From partitioned maps to permuted star thorn trees

Idea : merge vertices of the same block and cut all edges except one per block



Rules to merge vertices and choose which edges to keep:

1. Draw each vertex with its maximum as right-most edge and order them in decreasing order of their maxima (like in Foata's transform).



2. Merge vertices and keep the left-most edge.

Proposition 6. • *Injective mapping (ask for a demo of the inverse).*

- *Its image can be characterized using auxiliary graphs:*

P ⟵ R   Q ⇄ S   T ⟵ U

Using this characterization, one can check that the proportion of the image is $\frac{1}{N-p+1}$. □

Remarks

- Another simpler combinatorial proof has been found recently in [1].
- Analogue results for maps on locally orientable surfaces?

References

[1] R. Cori, M. Marcus, G. Schaeffer, *On the number of cycles of the product of two cyclic permutations*, Lattice paths, 2010.

[2] A. Morales and E. Vassilieva, *Bijective enumeration of bicolored maps of given vertex degree distribution*, DMTCS Proceedings (FPSAC), **AK**, 661–672, 2009.

[3] R. Stanley, *Two enumerative results on cycles of permutations*, arXiv:0901.2008, 2009.

[4] D. Zagier, *On the distribution of the number of cycles of elements in symmetric groups*, Nieuw Arch. Wisk., **13**:489—495, 1995.