Random Young diagrams and tableaux
Lecture 5: determinantal point processes

Valentin Féray

CNRS, Université de Lorraine

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What is a (discrete) determinantal point process?

Notation:

- E is a (discrete) countable set;
- $X \subseteq E$ is a **random** subset of E;
- $K : E \times E \to \mathbb{R}$ a function called *kernel*.

Definition

X is a determinantal point process (DPP) if, for any finite subset $A \subset E$, one has

$$\mathbb{P}(A \subseteq X) = \det\left[(K(x,y))_{a,b \in A}\right].$$

Many instances: eigenvalues of random matrices, edge set of uniform spanning tree of any graph, descent set of a uniform random permutation, random diagrams and tableaux...

Why are DPPs useful?

Consider a sequence of DPP X_n on a set E, with kernel K_n.

Assume that, for some α_n and β_n, we have: for all a, b in E

$$\lim \beta_n K_n(\alpha_n + \beta_n a, \alpha_n + \beta_n b) = K(a, b).$$

Then the normalized sets

$$\widetilde{X}_n = \{x : \alpha_n + \beta_n x \in X_n\}$$

converge to a DPP of kernel K.

Poissonized Plancherel measure is a DPP

Reminder: for a diagram λ, $D(\lambda)$ are the x-coordinate of the (middle) of its down steps.

Poissonized Plancherel measure with parameter θ:

$$\mathbb{P}_\theta(\lambda) = e^{-\theta}\theta^{|\lambda|}\frac{\dim(\lambda)}{|\lambda|!}.$$

This is a probability of the set of Young diagrams of all sizes, but if $\theta = n$, it ressembles the Plancherel measure on diagrams of size n.

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Theorem (Borodin, Okounkov, Olshanski, '00)

If λ has distribution $\mathbb{P}_\theta$, then $D(\lambda)$ is a DPP with kernel

$$K_\theta(a,b) := \sqrt{\theta}\,\frac{J_{a-\frac{1}{2}}(2\sqrt{\theta})J_{b+\frac{1}{2}}(2\sqrt{\theta}) - J_{a+\frac{1}{2}}(2\sqrt{\theta})J_{b-\frac{1}{2}}(2\sqrt{\theta})}{a-b},$$

where $J_\alpha(z) = \sum_{p\geq 0} \frac{(-1)^p}{p!\Gamma(p+\alpha+1)}\left(\frac{z}{2}\right)^{2p}$ *is known as a* Bessel function.

Interesting limits of the kernel

Lemma

Fix $\alpha \in [-2;2]$.

$$\lim_{\theta \to +\infty} K_\theta(\alpha\sqrt{\theta} + a, \alpha\sqrt{\theta} + b) \to \frac{\sin\left(\cos^{-1}\left(\frac{\alpha}{2}\right) \cdot k\right)}{\pi\, k}.$$

Consequence : around position $\alpha\sqrt{\theta}$, the random set $D(\lambda)$ looks like a DPP, called the *sine kernel*.

For $\alpha = 2$, the limiting kernel is 0, which means that descending steps have become rare.
We need another rescaling.

Interesting limits of the kernel

Lemma

Fix $\alpha \in [-2;2]$.

$$\lim_{\theta \to +\infty} K_\theta(2\sqrt{\theta} + \theta^{1/6} a, 2\sqrt{\theta} + \theta^{1/6} b) \to A(a,b),$$

for some function A, called Airy kernel*.*

Hence around the *edge* $x = 2\sqrt{\theta}$, after rescaling, the random set $D(\lambda)$ looks like another DPP, called the *Airy kernel*.

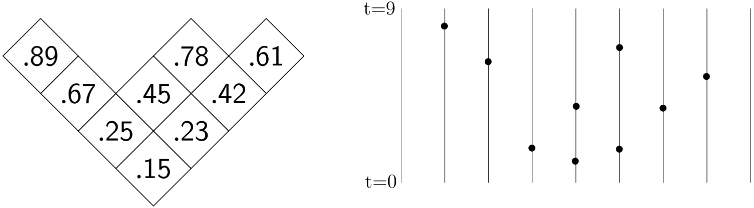
→ This lemma is a key step in the proof that the longest increasing subsequence in a random permutation has Tracy-Widom fluctuations.

Poissonized tableaux...

A Poissonized tableau T of shape λ is a filling of λ with numbers in $[0,1]$, with increasing rows and columns. We encode it as

$$M_T := \Big\{ (x(\square), T(\square)), \square \in \lambda \Big\},$$

which is a subset in $\mathbb{Z} \times [0,1]$



A Poissonized Young tableau T and the associated set M_T.

... are determinantal point processes

Theorem (Gorin, Rahman, '19)

Fix a diagram λ and let T be a uniform random Poissonized tableau of shape λ. Then the associated random subset M_T is determinantal with kernel

$$K_\lambda((x_1,t_1),(x_2,t_2)) = \mathbf{1}_{x_1>x_2,\, t_1<t_2} \frac{(t_1-t_2)^{x_1-x_2-1}}{(x_1-x_2-1)!} - \frac{1}{(2\mathrm{i}\pi)^2} \oint_{\gamma_z}\oint_{\gamma_w} \frac{F_\lambda(x_2+z)}{F_\lambda(x_1-1+w)} \frac{\Gamma(w)}{\Gamma(z+1)} \frac{(1-t_2)^z(1-t_1)^{-w}}{z-w+x_2-x_1+1}\, dw\, dz,$$

where

$$F_\lambda(u) := \Gamma(u+1) \prod_{i=1}^{\infty} \frac{u+i}{u-\lambda_i+i},$$

and γ_w and γ_z are well-chosen integration paths.

Some consequences

- local limits of staircase tableaux near the outer-diagonal (Gorin–Rahman '19, motivated by the study of random sorting networks);
- local limits of multi-rectangular tableaux near a point in the bulk (work in progress with J. Borga, C. Boutillier and P.-L. Méliot).