Random Young diagrams and tableaux
Exercises
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Exercise 1. Consider Gelfand mesure on partitions of n, defined by $\mathbb{P}_G(\lambda) = \dim(\lambda)/Z$, where Z is the number of involutions of n. What is the limit shape of λ, as n tends to $+\infty$?

Exercise 2. Let $\mathcal{SP}(n)$ be the set of set partitions of $[n]$ and take $q > 1$. Set partitions are seen as set of arcs (i, j), where i and j are *consecutive* elements in the same block. We let $d(\pi)$ be the number of arcs, $\mathrm{crs}(\pi)$ be the number of crossings and $\dim(\pi) = \sum_{(i,j) \text{ arc of } \pi}(j - i)$ be the sum of the length of the arcs.

We define a *superPlancherel* probability measure on $\mathcal{SP}(n)$ by the formula

$$\mathrm{SPl}_n(\pi) = \frac{1}{q^{\frac{n(n-1)}{2}}} \frac{(q-1)^{d(\pi)} \cdot q^{2\dim(\pi)-d(\pi)}}{q^{\mathrm{crs}(\pi)}}.$$

We refer to [DS18] for the proof that SPl_n is a probability measure and for its representation-theoretical significance.

Finally, with a set-partition π in $\mathcal{SP}(n)$, we associate a measure

$$\mu_\pi = \frac{1}{n} \sum_{(i,j) \text{ arc of } \pi} \delta_{(i/n, j/n)}.$$

This is a measure of total weight at most 1 on the triangle $\Delta := \{(x, y) : 0 \le x \le y \le 1\}$.

(a) Write $\mathrm{SPl}_n(\pi) = \exp\big(-n^2 \log(q) I(\mu_\pi) + O(n)\big)$ for some appropriate functional I.

(b) Find the maximizer of I on the set of measure on Δ satisfying: for all $a \le b$ in $[0, 1]$

$$\mu([a, b] \times [0, 1]) \le b - a \text{ and } \mu([0, 1] \times [a, b])$$

(c) Deduce a law of large numbers for (the measure associated to) a random set partition with law SPl_n.

Exercise 3. (a) Let T_n be a uniform random tableau of size n (the shape of T is not fixed). Fix $k \ge 1$ and a tableau U of shape μ of size k. Show that

$$\lim_{n\to+\infty} \mathbb{P}\big(T_n/[k] = U\big) = \frac{\dim(U)}{k!}, \tag{1}$$

where $T_n/[k]$ is the restriction of the tableau T_N to the boxes containing entries less or equal to k.

Hint: by RSK algorithm, T_n can be construct as the recording tableau of a uniform random *involution* in S_n. We shall admit that a uniform random involution has $o(n)$ fixed points with high probability.

(b) (Needs basic knowledge of representation theory) Let $\mu \subseteq \lambda$ be Young diagrams. Denoting $f^{\lambda/\mu}$ the number of Young tableaux of skew shape λ/μ, prove that

$$f^{\lambda/\mu} = \frac{1}{|\mu|!} \sum_{\sigma\in S_k} \chi^\lambda(\sigma)\chi^\mu(\sigma).$$

Let λ_n be a sequence of A-balanced shape (i.e. $\max(\lambda_1, \ell(\lambda) \le A\sqrt{n})$ for some A. Prove an analogue of (1), where T_n is a uniform tableau of shape λ_n (as above, the tableau U and its shape μ are however fixed).

(c) As above, let λ_n be a sequence of A-balanced shape, and T_n a uniform random tableau of shape λ_n. Let $k = k_n$ tends to $+\infty$, with $k \ll n$. Show that the shape of the truncation $T_n/[k]$ tends to the Logan–Shepp–Vershik–Kerov limit shape Ω.

Exercise 4. We consider the following element of the symmetric group algebra $\mathbb{C}[S_n]$:

$$C_{(2)} := \sum_{i<j} (i, j).$$

We recall that, since $C_{(2)}$ is central, for any irreductible representation V_λ, we have

$$\rho^\lambda(C_{(2)}) = X_\lambda \, \mathrm{Id}_{V_\lambda} \text{ , where } X_\lambda := \frac{\chi^\lambda(C_{(2)})}{\chi^\lambda(1)}.$$

Fix r even.

(a) Show that $\mathbb{E}_{Pl_n}[X_\lambda^r]$ is, up to a normalization factor, the number of factorizations of the identity as a product of r transpositions.

(b) Admitting that most such factorizations are obtained by taking $r/2$ disjoint factorizations, repeating each twice, and shuffling the output, find an asymptotic equivalent for $\mathbb{E}_{Pl_n}[X_\lambda^r]$.

(c) Deduce a limit theorem (convergence in distribution after normalization) for X_λ and for $R_3(\lambda)$, when λ is distributed with Plancherel measure of size n.

(We recall that if a sequence Y_n of r.v. satisfies that, for each r, $\mathbb{E}[Y_n^r] \to (2r-1)!!$, then Y_n converges in distribution to a standard Gaussian variable.)

Exercise 5. The goal of the exercise is to re-discover the formula for the Logan–Shepp–Vershik–Kerov limit shape Ω, starting from the values of its free cumulants $R_k(\Omega) = \delta_{k,2}$. For this we recall a few formulae:

- The compositional of the Cauchy transform $G(z)$ of a diagram is $K(u) = u^{-1} + \sum_k R_k u^{k-1}$.
- $G(z)$ is related to ω by

$$\log(z\, G(z)) = -\frac{1}{2} \int_{\mathbb{R}} \frac{\omega'(u) - \mathrm{sg}(u)}{z-u} du.$$

- Finally we have Stieltjes inversion formula: if $S_\rho(z) = \int_{\mathbb{R}} \frac{\rho(u)du}{z-u}$, with ρ continuous, then

$$\rho(x) = \lim_{\varepsilon \to 0^+} \frac{S_\rho(x - i\varepsilon) - S_\rho(x + i\varepsilon)}{2\pi\, i}.$$

Bibliographic note: Exercice 1 is based on [Mél11], where the second order asymptotics is also discussed. Exercise 2 is based on [DS18], Question 3.1 on [MMW02]. For Question 3.2, see [Sta03, DF19]. Exercise 4 is a particular case of a result of Hora [Hor98] and Kerov (written in details by Ivanov and Olshanski [IO02]), separately. Regarding exercise 5: a more involved computation to recover a formula for a limit shape, starting from the value of cumulants can be found in [Bia01].

References

[Bia01] P. Biane. Approximate factorization and concentration for characters of symmetric groups. *Int. Math. Res. Not.*, 2001(4):179–192, 2001.

[DF19] J. Dousse and V. Féray. Asymptotics for skew standard Young tableaux via bounds for characters. *Proc. Am. Math. Soc.*, 147(10):4189–4203, 2019.

[DS18] D. De Stavola. A Plancherel measure associated to set partitions and its limit. *Adv. Appl. Math.*, 92:73–98, 2018.

[Hor98] A. Hora. Central limit theorem for the adjacency operators on the infinite symmetric group. *Commun. Math. Phys.*, 195(2):405–416, 1998.

[IO02] V. Ivanov and G. Olshanski. Kerov's central limit theorem for the Plancherel measure on Young diagrams. In *Symmetric functions 2001: Surveys of developments and perspectives. Proceedings of the NATO Advanced Study Institute, Cambridge, UK, June 25–July 6, 2001*, pages 93–151. Dordrecht: Kluwer Academic Publishers, 2002.

[Mél11] P.-L. Méliot. Kerov's central limit theorem for Schur-Weyl and Gelfand measures (extended abstract). In *Proceedings of the 23rd international conference on formal power series and algebraic combinatorics, FPSAC 2011, Reykjavik, Iceland, June 13–17, 2011*, pages 669–680. Nancy: The Association. Discrete Mathematics & Theoretical Computer Science (DMTCS), 2011.

[MMW02] B. D. McKay, J. Morse, and H. S. Wilf. The distributions of the entries of Young tableaux. *J. Comb. Theory, Ser. A*, 97(1):117–128, 2002.

[Sta03] R. P. Stanley. On the enumeration of skew Young tableaux. *Adv. Appl. Math.*, 30(1-2):283–294, 2003.