Classes of Sum Decomposable Affine Permutations

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Erratum from PP 2018

I neglected to insert a key phrase in my very impressive slide titled *General result on growth rates of sequences*. There, I list several conditions and conclude that, under those conditions,

$$\textit{If } c_n^{(1)} \leqslant c_n^{(2)} \textit{ for all } n \textit{ and } c_n^{(1)} < c_n^{(2)} \textit{ for some } n\textit{, then } \operatorname{gr}(a_n^{(1)}) < \operatorname{gr}(a_n^{(2)}).$$

It was a very strong theorem.

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> *If* $c_n^{(1)} \leqslant c_n^{(2)}$ *for all* n *and* $c_n^{(1)} < c_n^{(2)}$ *for some* n*, then it is not necessarily the case that* $\mathrm{gr}(a_n^{(1)}) < \mathrm{gr}(a_n^{(2)})$.

It was a very ~~strong~~ wrong! theorem.

Affine permutations

- An affine permutation of size n is a bijection $\sigma\colon \mathbb{Z} \to \mathbb{Z}$ such that:
  (i) $\sigma(i+n) = \sigma(i) + n$ for all $i \in \mathbb{Z}$;
  (ii) $\sum_{i=1}^{n} \sigma(i) = \sum_{i=1}^{n} i$.

  Condition (ii) can also be written as $\sum_{i=1}^{n} (\sigma(i) - i) = 0$.
- The affine permutations of size n are an infinite Coxeter group.
- An affine permutation σ of size n is determined by its values on $\{1, \ldots, n\}$, and taking these values mod n results in an (ordinary) permutation.
- An affine permutation σ of size n is bounded if $|\sigma(i) - i| < n$ for all i.

$|\mathbf{AvBA}(321)|$ versus $|\mathbf{AvBA}(312)|$

- Given a set R of permutations, $\mathrm{AvA}(R)$ is the class of affine permutations avoiding every element of R.
  - For each $n \geqslant 2$, $\mathrm{AvA}_n(321)$ is infinite.

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  - $\mathrm{AvBA}_n(321)$ is finite, but we have not found an exact enumeration.
- However, $|\mathrm{AvA}_n(312)| = |\mathrm{AvBA}_n(312)| = \frac{1}{2}\binom{2n}{n}$.

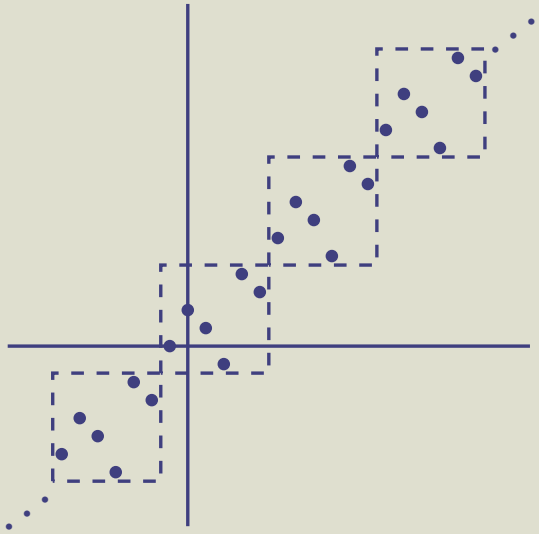
I. **Decomposability and the infinite oscillation**

II. Enumeration of decomposable affine permutations

- An affine permutation σ of size n is sum decomposable if there is $k \in \mathbb{Z}$ such that

$$i > k \Leftrightarrow \sigma(i) > k.$$

- This is equivalent to the condition that there is an (ordinary) permutation π of size n such that σ has the form $(\cdots \oplus \pi \oplus \pi \oplus \cdots)$, possibly shifted diagonally.



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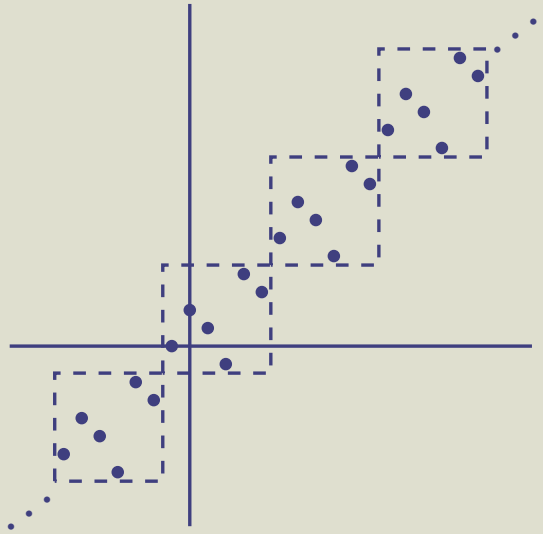


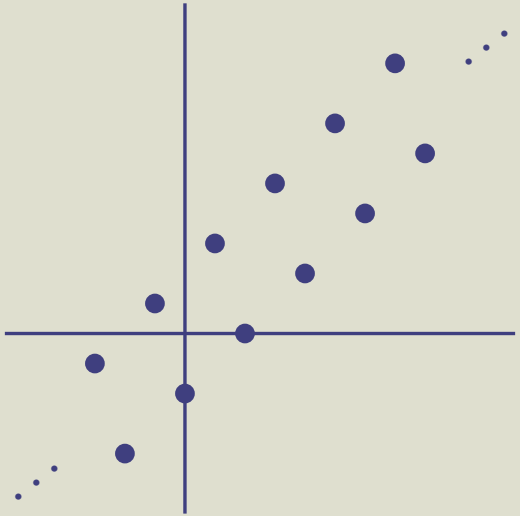
When does an affine permutation class consist only of decomposable elements?

The infinite oscillation

- The infinite oscillation is the affine permutation

$$\mathcal{O} = \ldots 30\,52\,74\,96 \ldots$$

  - $\mathcal{O}$ has been studied (e.g. by Albert, Brignall, and Vatter) because it can be used to construct infinite antichains.



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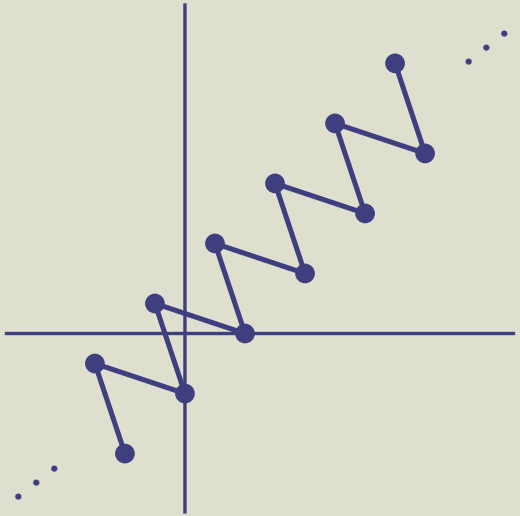

- $\mathcal{O}$ is indecomposable!

When is an affine permutation class decomposable?

Madras and T. 2019+:

- An affine permutation is decomposable if and only if it avoids $\mathcal{O}$.
- An affine permutation class is decomposable if and only if $\mathcal{O}$ is not in it (if and only if it is a subset of $\mathrm{Av}\mathrm{A}(\mathcal{O})$).
- $\mathrm{Av}\mathrm{A}(R)$ is decomposable if and only if there is an element of R that is contained in $\mathcal{O}$.
  - Note that the indecomposable permutations contained in $\mathcal{O}$ are the finite oscillations

$$1, 21, 312, 231, 3142, 2413, 31524, 24153, 315264, 241635, \ldots$$

I. Decomposability and the infinite oscillation

II. **Enumeration of decomposable affine permutations**

Infinite direct sums

Assume $\mathcal{C}$ is sum closed.

- Let $\oplus\mathcal{C}$ denote the set of affine permutations obtained by taking infinite direct sums of permutations in $\mathcal{C}$ (possibly with a diagonal shift).
  - By our theorem, $\oplus\mathrm{Av}(R) = \mathrm{AvA}(R \cup \{\mathcal{O}\})$.
- Let $\mathcal{C}^{\not\oplus}$ denote the set of (ordinary) permutations in $\mathcal{C}$ that are indecomposable.
- Madras and T. 2019+: If $A(x)$ and $\widetilde{A}(x)$ are the generating functions for $\mathcal{C}$ and $\oplus\mathcal{C}$ respectively, then

$$\widetilde{A}(x) = x\tfrac{d}{dx}\log(A(x)) = \frac{xA'(x)}{A(x)}.$$

- Madras and T. 2019+: $\max\left\{|\mathcal{C}_n|,\, n|\mathcal{C}^{\not\oslash}|\right\} \leqslant |\oplus\mathcal{C}_n| \leqslant n|\mathcal{C}_n|$.
  - In contrast, $|\mathrm{AvAB}_n(321)| \sim (n^2/2)|\mathrm{Av}_n(321)|$.
  - If an element of R is contained in $\mathcal{O}$, then this shows that $\mathrm{gr}(\oplus\mathrm{Av}(R)) = \mathrm{gr}(\mathrm{Av}(R))$ and it answers the question of how much bigger $\oplus\mathrm{Av}_n(R)$ is than $\mathrm{Av}_n(R)$.

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Supercritical and subcritical are technical conditions on the generating function of $\mathcal{C}$ as it relates to the generating function of $\mathcal{C}^{\not\oplus}$. (See Flajolet and Sedgewick, *Analytic Combinatorics*.)

- Supercritical implies that the number of blocks of almost all permutations in $\mathcal{C}_n$ is $\sim \alpha n$ for constant $\alpha \in (0,1)$.
  - In this situation, $\mathcal{C}^{\not\oplus}$ has a smaller growth rate than $\mathcal{C}$.
  - Madras and T. 2019+: $|\oplus\mathcal{C}_n| \sim \alpha^{-1}|\mathcal{C}_n| \sim L^n$.
- Subcritical implies that $|\mathcal{C}_n^{\not\oplus}| \sim \beta|\mathcal{C}_n|$ for some $\beta \in (0,1)$.
  - In this situation, $\mathcal{C}^{\not\oplus}$ has the same growth rate as $\mathcal{C}$.
  - Bóna 2019+: As $n \to \infty$, $\mathrm{Av}(\pi)$ is subcritical* for almost all $\pi \in S_n^{\not\oplus}$.
  - Madras and T. 2019+: $|\oplus\mathcal{C}_n| \sim \beta^{1/2} n|\mathcal{C}_n|$.

Examples

R	type	$\lvert Av_n(R)\rvert$	$\lvert AvA_n(R)\rvert$
$312, 231$	supercrit.	2^{n-1}	$2^n - 1 \sim 2\lvert Av_n(R)\rvert$
312	subcrit.	$\frac{1}{n+1}\binom{2n}{n}$	$\frac{1}{2}\binom{2n}{n} \sim \frac{n}{2}\lvert Av_n(R)\rvert$
$3142, 2413$	subcrit.	$\sim k\, n^{-3/2}\, \rho^n$	$\sim \frac{n}{\sqrt{2}}\lvert Av_n(R)\rvert$
3142	subcrit.	$\sim k\, n^{-5/2}\, 8^n$	$\sim \frac{27}{32} n\lvert Av_n(R)\rvert$

$(\rho = 3 + 2\sqrt{2} \approx 5.828)$