

On the Equidistribution of MAJ and BAST

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joint with Joanna N. Chen, T. Hua, V. Vajnovszki

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Plan of the talk

Introduction

Main results

Sketch of one bijective proof

Example

Introduction

- ▶ Mahonian statistics and vincular patterns
- ▶ q -Stirling numbers of the 2nd kind: two versions
- ▶ RGF and four “inversion-like” statistics
- ▶ How are they connected?

Mahonian statistics

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Let $\mathfrak{S}_n$ be the symmetric group on n letters and $\mathbb{N}$ the nonnegative integers. A function $f : \mathfrak{S}_n \rightarrow \mathbb{N}$ is called a *permutation statistic*.

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$$\sum_{\pi \in \mathfrak{S}_n} q^{\text{INV}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} q^{\text{MAJ}(\pi)} = 1 \cdot (1 + q) \cdots (1 + q + \cdots + q^{n-1}) = n!_q.$$

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O. Rodriguez, *Note sur les inversions, ou dérangements produits dans les permutations*, J. de Math., 1839.



P. A. MacMahon, *Combinatory Analysis*, vols. 1 and 2, Cambridge Univ. Press, 1915.

Permutation Patterns

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A *pattern* p in a permutation π is a subsequence of π that has the same pairwise comparisons as p .

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S. Kitaev, *Patterns in permutations and words*, Springer Science & Business Media, 2011.



E. Babson and E. Steingrímsson, *Generalized permutation patterns and a classification of the Mahonian statistics*, Sémin. Lothar. Combin, 2000.

Permutation symmetries

Recall three basic permutation symmetries. Reversal, r , complement, c , and the composition $rc = r \circ c$.

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Fun exercise: write INV and MAJ using vincular patterns. The number of ways of combining one of the 2-patterns 12 and 21 and three 3-patterns with precisely one partial underlining each is

$$2 \cdot \binom{14}{3} = 728, \quad 728/4 = 182$$

if we count modulo symmetries.

Classification of the Mahonian 3-functions

1	<u>21</u>	+	<u>132</u>	+	<u>132</u>	+	<u>213</u>	
2	<u>21</u>	+	<u>132</u>	+	<u>132</u>	+	<u>231</u>	
3	<u>21</u>	+	<u>132</u>	+	<u>231</u>	+	<u>231</u>	
4	<u>21</u>	+	<u>231</u>	+	<u>231</u>	+	<u>312</u>	MAD
5	<u>21</u>	+	<u>132</u>	+	<u>213</u>	+	<u>321</u>	STAT
6	<u>21</u>	+	<u>132</u>	+	<u>213</u>	+	<u>321</u>	
7	<u>21</u>	+	<u>132</u>	+	<u>231</u>	+	<u>321</u>	MAK
8	<u>21</u>	+	<u>132</u>	+	<u>231</u>	+	<u>321</u>	MAJ
9	<u>21</u>	+	<u>132</u>	+	<u>312</u>	+	<u>321</u>	MAKL
10	<u>21</u>	+	<u>132</u>	+	<u>312</u>	+	<u>321</u>	STAT'
11	<u>21</u>	+	<u>132</u>	+	<u>231</u>	+	<u>231</u>	
12	<u>21</u>	+	<u>132</u>	+	<u>312</u>	+	<u>321</u>	STAT''
13	<u>21</u>	+	<u>231</u>	+	<u>312</u>	+	<u>312</u>	
14	<u>21</u>	+	<u>231</u>	+	<u>312</u>	+	<u>321</u>	INV

All Mahonian 3-functions up to basic symmetries.

Classification of the Mahonian 3-functions

$$5 \quad \underline{21} \quad + \quad \underline{132} \quad + \quad \underline{213} \quad + \quad \underline{321} \quad \text{STAT}$$

$$8 \quad \underline{21} \quad + \quad \underline{132} \quad + \quad \underline{231} \quad + \quad \underline{321} \quad \text{MAJ}$$

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on words with same length and alphabet.

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- ▶ Fu-Hua-Vajnovszki (2019):

$$(\text{F}, \text{des}, \text{Id}, \text{MAJ}, \text{STAT}) \sim (\text{F}, \text{des}, \text{Id}, \text{STAT}, \text{MAJ})$$

on the rearrangement class $R(w)$ for any word w .

Two q -Stirling numbers

$$S_q(n, k) = \begin{cases} q^{k-1}S_q(n-1, k-1) + [k]_q S_q(n-1, k) & \text{if } 0 < k \leq n \\ 1 & \text{if } n = k = 0 \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

and

$$\tilde{S}_q(n, k) = \begin{cases} \tilde{S}_q(n-1, k-1) + [k]_q \tilde{S}_q(n-1, k) & \text{if } 0 < k \leq n \\ 1 & \text{if } n = k = 0 \\ 0 & \text{otherwise,} \end{cases} \quad (2)$$

where

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First studied by Carlitz and Gould.



L. Carlitz, *q -Bernoulli numbers and polynomials*, Duke Math., (1948).



H. W. Gould, *The q -Stirling numbers of the first and second kind*, Duke Math., (1961).

Set partitions and restricted growth functions (RGF)

Given a partition of $[n]$, order the blocks so that their smallest numbers form an increasing sequence, say $\pi = B_1/B_2/\dots$. Then function $f : [n] \rightarrow [n]$ is defined by requiring $i \in B_{f(i)}$. Denoted as $\mathcal{RG}(n, k)$ and $\mathcal{RG}(n)$.

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$$\begin{aligned} &6, 7 / 1, 2, 5 / 3, 4, 8 \Rightarrow 1, 2, 5 / 3, 4, 8 / 6, 7 \\ \Rightarrow &\begin{array}{cccccccc} f(1) & f(2) & f(3) & f(4) & f(5) & f(6) & f(7) & f(8) \\ 1 & 1 & 2 & 2 & 1 & 3 & 3 & 2 \end{array} \end{aligned}$$

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- $f(1) = 1, f(i) \leq 1 + \max\{f(1), f(2), \dots, f(i-1)\}$ for $1 < i \leq n$.

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G. Hutchinson, *Partitioning algorithms for finite sets*, Communications of the ACM, 1963.



S. C. Milne, *Restricted growth functions and incidence relations of the lattice of partitions of an n -set*, Adv. in Math., 1977.



S. C. Milne, *A q -analog of restricted growth functions, Dobinski's equality, and Charlier polynomials*, Trans. of the AMS, 1978.

Four “inversion-like” statistics

Given any word $w = w_1 w_2 \cdots w_n$, let $\text{Asc}(w) = \{i : w_i < w_{i+1}\}$.

coordinate

vector

global

$$\text{st}_i(w)$$

$$\mathbf{st}(w) := (\text{st}_1(w) \cdots \text{st}_n(w))$$

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$$\text{L}(w) = \{i \in [n] : w_i \text{ is the leftmost occurrence of } w_i\},$$

$$\text{R}(w) = \{i \in [n] : w_i \text{ is the rightmost occurrence of } w_i\}.$$

For $i \in [n]$, let

$$\text{lb}_i(w) = \# \{j \in \text{L}(w) : j < i \text{ and } w_j > w_i\},$$

$$\text{ls}_i(w) = \# \{j \in \text{L}(w) : j < i \text{ and } w_j < w_i\},$$

$$\text{rb}_i(w) = \# \{j \in \text{R}(w) : j > i \text{ and } w_j > w_i\},$$

$$\text{rs}_i(w) = \# \{j \in \text{R}(w) : j > i \text{ and } w_j < w_i\}.$$

As an example, for $w = 12134243221435322$, we see that

$$L(w) = \{1, 2, 4, 5, 14\},$$

$$R(w) = \{11, 12, 14, 15, 17\},$$

$$lb(w) = (0\ 0\ 1\ 0\ 0\ 2\ 0\ 1\ 2\ 2\ 3\ 0\ 1\ 0\ 2\ 3\ 3),$$

$$ls(w) = (0\ 1\ 0\ 2\ 3\ 1\ 3\ 2\ 1\ 1\ 0\ 3\ 2\ 4\ 2\ 1\ 1),$$

$$rb(w) = (4\ 3\ 4\ 2\ 1\ 3\ 1\ 2\ 3\ 3\ 4\ 1\ 1\ 0\ 0\ 0\ 0),$$

$$rs(w) = (0\ 1\ 0\ 2\ 3\ 1\ 3\ 2\ 1\ 1\ 0\ 2\ 1\ 2\ 1\ 0\ 0).$$

Consequently, we have $lb(w) = 20$, $ls(w) = 27$, $rb(w) = 32$ and $rs(w) = 20$.

Theorem (Milne, Wachs-White)

For $n \geq k \geq 1$, we have

$$\sum_{w \in \mathcal{RG}(n,k)} q^{\text{ls}(w)} = \sum_{w \in \mathcal{RG}(n,k)} q^{\text{rb}(w)} = S_q(n, k), \quad (3)$$

and

$$\sum_{w \in \mathcal{RG}(n,k)} q^{\text{lb}(w)} = \sum_{w \in \mathcal{RG}(n,k)} q^{\text{rs}(w)} = \tilde{S}_q(n, k). \quad (4)$$



S. C. Milne, *A q -analog of restricted growth functions, Dobinski's equality, and Charlier polynomials*, Trans. of the AMS, 1978.



M. Wachs and D. White, *p, q -Stirling numbers and partition statistics*, J. Combin. Theory Ser. A, 1991.

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Claesson observed in 2001, that there is a natural bijection

$$\pi = B_1/B_2/\dots/B_k \longrightarrow p = B_k B_{k-1} \dots B_1 \in \mathfrak{S}_n(\underline{132}),$$

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$$\begin{array}{rclclcl} \text{STAT} = & \underline{21} & + & \underline{132} & + & \underline{213} & + & \underline{321} \\ \text{MAJ} = & \underline{21} & + & \underline{132} & + & \underline{231} & + & \underline{321} \\ \text{BAST} = \text{STAT}^{\text{rc}} = & \underline{21} & + & \underline{132} & + & \underline{213} & + & \underline{321} \end{array}$$



A. Claesson, *Generalized Pattern Avoidance*, Europ. J. Combinatorics, 2001.

Main results

- ▶ New interpretations for $S_q(n, k)$ and $\tilde{S}_q(n, k)$
- ▶ Joint equidistribution over $\mathcal{RG}(n)$
- ▶ Joint equidistribution over $\mathfrak{S}_n(132)$
- ▶ A further relation between STAT and BAST
- ▶ Extension to ordered set partitions
- ▶ Extension to ordered multiset partitions and the Delta conjecture

Main results I

Let $\text{pro}(w)$ be the **p**osition of the **r**ightmost **o**ne in w . For $i \in [n]$, let

$$\text{vls}_i(w) := \begin{cases} \text{ls}_i(w) - 1 & \text{if } i \notin R(w) \text{ and } i > \text{pro}(w) \\ \text{ls}_i(w) + 1 & \text{if } i \in R(w) \text{ and } i < \text{pro}(w) \\ \text{ls}_i(w) & \text{otherwise,} \end{cases}$$

$$\text{vrs}_i(w) := \begin{cases} \text{rs}_i(w) + 1 & \text{if } i \notin R(w) \text{ and } i > \text{pro}(w) \\ \text{rs}_i(w) - 1 & \text{if } i \in R(w) \text{ and } i < \text{pro}(w) \\ \text{rs}_i(w) & \text{otherwise.} \end{cases}$$

Main results I

Let $\text{pro}(w)$ be the position of the rightmost one in w . For $i \in [n]$, let

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Theorem (1)

For $n \geq k \geq 1$, we have

$$\sum_{w \in \mathcal{RG}(n,k)} q^{\text{vls}(w)} = S_q(n, k), \quad (5)$$

and

$$\sum_{w \in \mathcal{RG}(n,k)} q^{\text{vrs}(w)} = \tilde{S}_q(n, k). \quad (6)$$

Main results II

Theorem (2)

Statistics $(L, \text{Asc}, \text{ls})$ and $(L, \text{Asc}, \text{vls})$ have the same joint distribution on $\mathcal{RG}(n)$ for all $n \geq 1$.

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Corollary (1)

Statistics $(\text{Db}, \text{Id}, \text{MAJ})$ and $(\text{Db}, \text{Id}, \text{BAST})$ have the same joint distribution on $\mathfrak{S}_n(\underline{132})$ for all $n \geq 1$.

Statistics $(\text{Db}, \text{Id}, \underline{213})$ and $(\text{Db}, \text{Id}, \underline{231})$ have the same joint distribution on $\mathfrak{S}_n(\underline{132})$ for all $n \geq 1$.

$$\begin{aligned}\text{Des}(p) &= \{i : p_i > p_{i+1}\}, \quad \text{Id}(p) = \text{Des}(p^{-1}), \\ \text{Db}(p) &= \{p_1\} \cup \{p_{i+1} : p_i > p_{i+1}, 1 \leq i < n\}.\end{aligned}$$

Main results III

Define the first letter $F(\pi) := \pi_1$, and the ending letter $E(\pi) := \pi_n$.

Theorem (4)

Statistics $(F, E, \text{des}, \text{Id}, \text{STAT}, \text{BAST})$ and $(F, E, \text{des}, \text{Id}, \text{BAST}, \text{STAT})$ have the same joint distribution on $\mathfrak{S}_n$ for all $n \geq 1$.

Main results IV

Lemma (Zeng-Zhang, 1994)

For all n and k with $0 \leq k \leq n$ we have

$$[k]_q! \cdot S_q(n, k) = \sum_i q^{k(k-i)} \cdot \begin{bmatrix} n-i \\ k-i \end{bmatrix}_q \cdot A_q(n, i-1),$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \frac{[n]_q!}{[k]_q! [n-k]_q!} & \text{for } n \geq k \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

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Theorem (5)

For $n \geq k \geq 1$, we have

$$\sum_{w \in \mathcal{URG}(n, k)} q^{\text{bmajMIL}(w)} = \sum_{w \in \mathcal{URG}(n, k)} q^{\text{bmajBAST}(w)} = [k]_q! \cdot S_q(n, k).$$

Main results V

The Valley Version of “Delta Conjecture” asserts the following combinatorial formula for the quasisymmetric function

$$\Delta'_{e_k} e_n = \text{Val}_{n,k}(x; q, t) := \{z^{n-k-1}\} \left[\sum_{P \in \mathcal{LD}_n} q^{\text{dinv}(P)} t^{\text{area}(P)} \prod_{i \in \text{Val}(P)} (1 + z/q^{d_i(P)+1}) x^P \right]$$

Theorem (6)

For all $n > k \geq 0$, we have

$$\begin{aligned} q^{\binom{k+1}{2}} \text{Val}_{n,k}(x; q, 0) &= q^{\binom{k+1}{2}} \sum_{\beta \models_0 n} \sum_{\mu \in \mathcal{OP}_{\beta, k+1}} q^{\text{inv}(\mu)} x^\beta \\ &= \sum_{\beta \models_0 n} \sum_{\mu \in \mathcal{OP}_{\beta, k+1}} q^{\text{bmajMIL}(\mu)} x^\beta = \sum_{\beta \models_0 n} \sum_{\mu \in \mathcal{OP}_{\beta, k+1}} q^{\text{bmajBAST}(\mu)} x^\beta. \end{aligned}$$



J. Haglund, J. Remmel, A. Wilson, *The delta conjecture*, Trans. Am. Math. Soc. 370.6 (2018): 4029–4057.



A. Garsia, J. Haglund, J. B. Remmel, M. Yoo, *A proof of the Delta Conjecture when $q = 0$* , Ann. Comb. 23.2 (2019): 317–333.

Robinson-Schensted-Knuth (RSK) algorithm

$$\pi \xrightarrow{RSK} (P_\pi, Q_\pi) \xrightarrow{RSK^{-1}} \pi$$

where RSK is a bijection between $\mathfrak{S}_n$ and the set of ordered pairs of n -cell standard Young tableaux (SYT) with the same shape.

We usually call P_π the *insertion tableau* and Q_π the *recording tableau* of π .

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Lemma

For any $\pi \in \mathfrak{S}_n$,

$$\begin{aligned} \text{Id}(j(\pi)) &= \text{Id}(\pi), \text{ and} \\ \text{Des}(j(\pi)) &= \{n - k : k \in \text{Des}(\pi)\}. \end{aligned}$$



D. Foata and M. P. Schützenberger, *Major index and inversion number of permutations*, Mathematische Nachrichten, 1978.

Involution ψ proving Theorem (4)

ϕ builds on j such that for any $\pi \in \mathfrak{S}_n$,

$$(\mathbf{F}, \text{des}, \text{Id}, \text{MAJ}, \text{STAT})\pi = (\mathbf{F}, \text{des}, \text{Id}, \text{STAT}, \text{MAJ})\phi(\pi).$$

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Given a word $w = w_1 w_2 \cdots w_n$ of length n , we define

$$\gamma^+(w) = u_1 u_2 \cdots u_n, \text{ where } u_j = \begin{cases} w_j + n & \text{if } w_j < w_n, \\ w_j & \text{otherwise.} \end{cases}$$

$$\gamma^-(w) = u_1 u_2 \cdots u_n, \text{ where } u_j = \begin{cases} w_j - n & \text{if } w_j > n, \\ w_j & \text{otherwise.} \end{cases}$$

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If $\pi_1 > \pi_n$, then we let

$$\psi(\pi) = \gamma^-(\phi(\text{pre}_{n-1}(\gamma^+(\pi))) \cdot \pi_n),$$

where $w \cdot u$ denotes the concatenation of two words w and u .

If $\pi_1 < \pi_n$, then we let

$$\psi(\pi) = r(\psi(r(\pi))),$$

where r is the reversal.

Example

Given $\pi = 10\ 9\ 1\ 2\ 8\ 6\ 4\ 3\ 7\ 5$, we get $\psi(\pi) = 10\ 6\ 4\ 1\ 9\ 8\ 2\ 3\ 7\ 5$, as follows.

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1	2	3
4		

P_{1243}

1	2	3
4		

Q_{1243}

1	3	4
2		

P_{2134}

1	3	4
2		

Q_{2134}

1	2
3	
4	

P_{4312}

1	4
2	
3	

Q_{4312}

1	4
2	
3	

P_{3421}

1	2
3	
4	

Q_{3421}

Future work

Study MAJ v.s. STAT or v.s. other Mahonian statistics over other pattern-avoiding subsets of $\mathfrak{S}_n$.

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N. Amini, *Equidistributions of Mahonian statistics over pattern avoiding permutations*, Electron. J. Combin. 25(1) (2018): P1.7.



