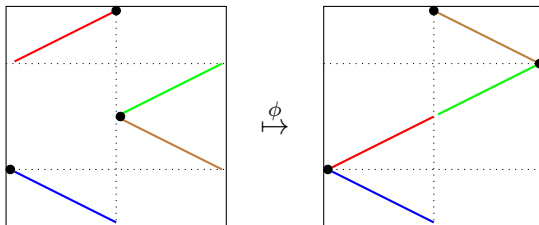


Cyclic Schur-Positive Permutation Sets

Jonathan Bloom (Lafayette), Sergi Elizalde (Dartmouth),
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Permutation Patterns, Zürich, June 2019

Descents and cyclic descents of permutations

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Introduced by Cellini ['95];
studied by Dilks-Petersen-Stembridge ['09] and others.

Descents and cyclic descents of permutations

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$$\pi = 31542 :$$

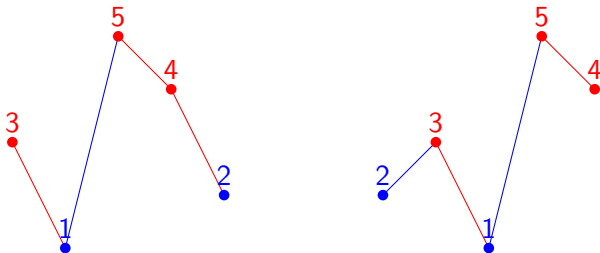
$$\sigma = 23154 :$$

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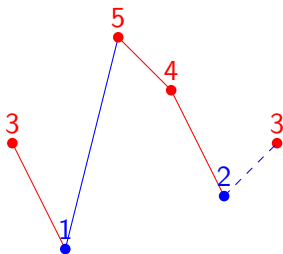
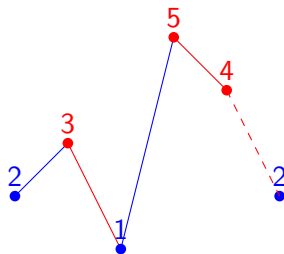


Descents and cyclic descents of permutations

Example

$\pi = \textcolor{red}{3}\textcolor{blue}{1}\textcolor{red}{5}\textcolor{blue}{4}\textcolor{red}{2}$: $\text{Des}(\pi) = \{1, 3, 4\}$, $\text{cDes}(\pi) = \{1, 3, 4\}$.

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Example

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 6 & \\ \hline 5 & 7 & \\ \hline \end{array} \in \text{SYT}((4, 3, 2)/(1, 1))$$

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Problem 1:

Define a **cyclic descent set** for SYT of any skew shape λ/μ .

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- (extension) $\text{cDes}(T) \cap [n-1] = \text{Des}(T)$,
- (equivariance) $\text{cDes}(p(T)) = p(\text{cDes}(T))$,
- (non-Escher) $\emptyset \subsetneq \text{cDes}(T) \subsetneq [n]$.

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Examples

- $\mathcal{T} = \mathfrak{S}_n$, with Cellini's cyclic descent set and $\mathbb{Z}_n$ -action by cyclic rotation.

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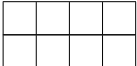
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- $\mathcal{T} = \text{SYT}(r^m)$  [Rhoades '10].

A non-Escher property



"Ascending and Descending", M. C. Escher

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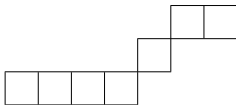


“Ascending and Descending”, M. C. Escher
The paradox of $\text{cDes}(\pi) = \emptyset$ or $\text{cDes}(\pi) = [n]$.

Special skew shapes

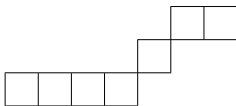
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1. Skew shapes avoiding  are called **horizontal strips**

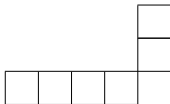


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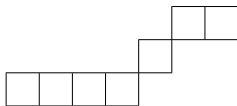


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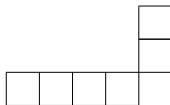


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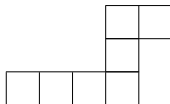
1. Skew shapes avoiding  are called **horizontal strips**



2. Skew shapes avoiding  and  are called **strips**



3. Skew shapes avoiding  are called **ribbons**



Example

1. For a SYT of a **horizontal strip** shape let

$$\text{cDes}(T) := \{i \in [n] : i+1 \bmod n \text{ is in a lower row than } i \text{ in } T\}.$$

$$T = \begin{array}{|c|c|} \hline & 3 & 4 \\ \hline 1 & 2 & \\ \hline \end{array} \quad 1 + T = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 3 & \\ \hline \end{array}$$

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- For a SYT T of a **strip** shape, let $\text{cDes}(T) := \text{Des}(T) \sqcup \{n\}$ iff either 1 is in a lower row than n , or 1 and n are in the same vertical component.

$$T = \begin{array}{|c|c|} \hline & 2 & 3 \\ \hline 1 & & \\ \hline 4 & & \\ \hline \end{array} \quad 1 + T = \begin{array}{|c|c|} \hline & 3 & 4 \\ \hline 1 & & \\ \hline 2 & & \\ \hline \end{array}$$

$$\text{cDes}(T) = \{3, 4\}, \quad \text{cDes}(1 + T) = \{4, 1\}.$$

Example (con't)

3. Two row shapes

The only cyclic descent map for $\lambda = (3, 2)$ is given by

<table><tr><td>1</td><td>3</td><td>4</td></tr><tr><td>2</td><td>5</td><td></td></tr></table>	1	3	4	2	5		<table><tr><td>1</td><td>2</td><td>5</td></tr><tr><td>3</td><td>4</td><td></td></tr></table>	1	2	5	3	4		<table><tr><td>1</td><td>3</td><td>5</td></tr><tr><td>2</td><td>4</td><td></td></tr></table>	1	3	5	2	4		<table><tr><td>1</td><td>2</td><td>4</td></tr><tr><td>3</td><td>5</td><td></td></tr></table>	1	2	4	3	5		<table><tr><td>1</td><td>2</td><td>3</td></tr><tr><td>4</td><td>5</td><td></td></tr></table>	1	2	3	4	5	
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Theorem (Adin-Reiner-Roichman '17)

1. *There **exists** a cyclic descent extension on $\text{SYT}(\lambda/\mu)$ iff λ/μ is not a connected ribbon.*

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The proof is algebraic. A combinatorial proof was given by Huang.

Schur-positive permutation sets

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Recall that

$$\text{SYM}_n \subseteq \text{QSYM}_n.$$

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Problem 2 [Gessel-Reutenauer '93]:

Find Schur-positive subsets in $\mathfrak{S}_n$.

Problem 3 [Sagan-Woo, PP '15]:

Find Schur-positive pattern avoiding classes in $\mathfrak{S}_n$.

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For $J \subseteq [n-1]$, let $\mathbf{x}^J := \prod_{i \in J} x_i$.

Theorem (Gessel)

A subset $A \subseteq \mathfrak{S}_n$ is Schur-positive if and only if there exist nonnegative integers $(m_{\lambda/\mu})_{\lambda/\mu \vdash n}$ such that

$$\sum_{\pi \in A} \mathbf{x}^{\text{Des}(\pi)} = \sum_{\lambda/\mu \vdash n} m_{\lambda/\mu} \sum_{T \in \text{SYT}(\lambda/\mu)} \mathbf{x}^{\text{Des}(T)}.$$

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Definition A subset $A \subseteq \mathfrak{S}_n$ is **cyclic Schur-positive (cSp)** if there exists a collection of nonnegative integers $(m_{\lambda/\mu})_{\lambda/\mu \vdash n}$ such that

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Most known Schur-positive sets of permutations are not cSp.

Main Results

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Theorem

For every Schur-positive set $A \subseteq \mathfrak{S}_{n-1}$, the set $AC_n \subseteq \mathfrak{S}_n$ is cSp .

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It follows that the set $C_n D_{n-1,J}^{-1}$ is cSp, thus, Schur-positive, providing an affirmative solution to Elizalde-R Conjecture [PP '15].

Pattern avoidance

The set of **arc permutations**

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Proof. $\mathfrak{S}_{n-1}(132, 312) = \bigsqcup_{i=0}^{n-2} D_{n-1,[i]}^{-1}.$



Pattern avoidance

Theorem

For every $n \geq 1$,

$$\sum_{\pi \in \mathcal{A}_n} \mathbf{x}^{\text{cDes}(\pi)} = \sum_{k=1}^{n-1} \sum_{T \in \text{SYT}((1^k) \oplus (n-k-1) \oplus (1))} \mathbf{x}^{\text{cDes}(T)}.$$

Pattern avoidance

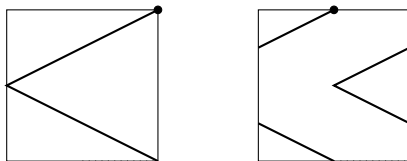


Figure: Horizontal rotation of $\mathfrak{S}_{n-1}(132, 312)$

Pattern avoidance

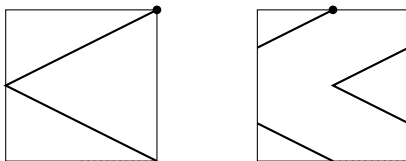


Figure: Horizontal rotation of $\mathfrak{S}_{n-1}(132, 312)$

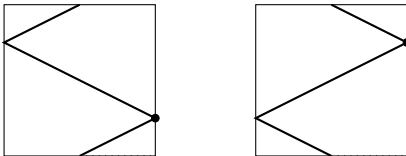


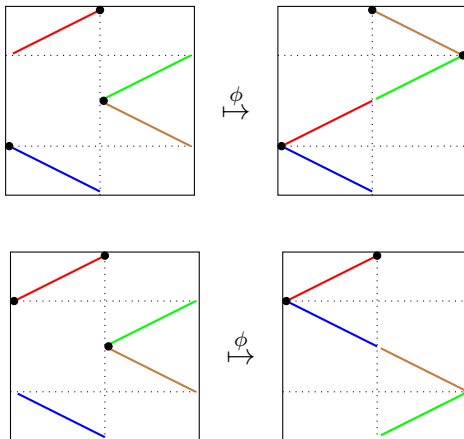
Figure: Vertical rotation of $\mathfrak{S}_{n-1}(132, 312)$

A pictorial description of a cDes-preserving bijection

$$\phi : \mathfrak{S}_{n-1}(132, 312)C_n \rightarrow C_n\mathfrak{S}_{n-1}(132, 312) = \mathcal{A}_n.$$

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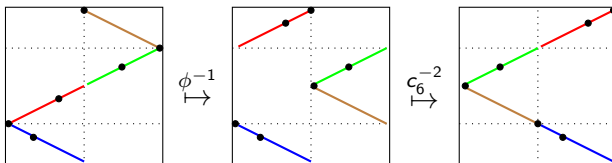
A cDes-preserving bijection from arc permutation to SYT

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$$\pi = 213645 \in \mathcal{A}_6 \longrightarrow \begin{array}{ccccccc} & & & & & & 4 \\ & & & & & 3 & 5 & 6 \\ & & & & 1 & & & \\ & & & 2 & & & & \end{array}$$

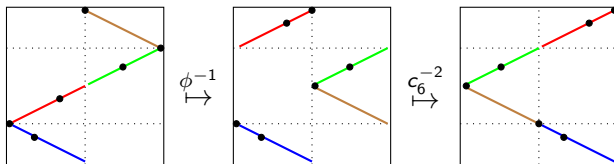
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$$\mapsto T = \begin{array}{|c|c|c|c|} \hline & & & 6 \\ \hline 1 & 2 & 5 & \\ \hline 3 & & & \\ \hline 4 & & & \\ \hline \end{array} \mapsto -2 + T = \begin{array}{|c|c|c|c|} \hline & & & 4 \\ \hline 3 & 5 & 6 & \\ \hline 1 & & & \\ \hline 2 & & & \\ \hline \end{array}.$$

Thank You !