

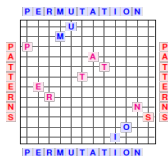
Baxter-like Patterns Avoiding

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Permutation Patterns 2019



Baxter-like patterns

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 - *Semi-Baxter permutations* = $\text{Av}(2-41-3)$ Bouvel, Guerrini, Rechnitzer, R. (2016);
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Baxter-like patterns

- Collect classical and recent results, preliminary investigations and open problems on permutations avoiding one to four *Baxter-like patterns*;
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 - recursive growth of $\text{Av}(\mathcal{E})$ and the corresponding generating tree $\Omega_{\mathcal{E}}$;
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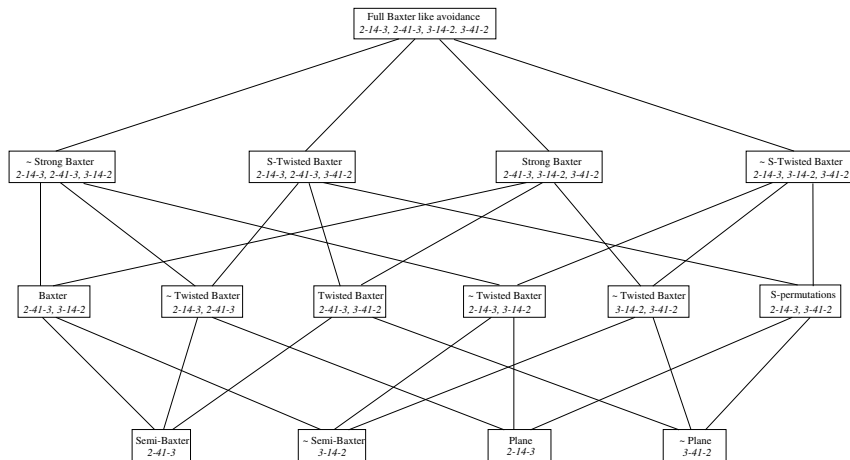
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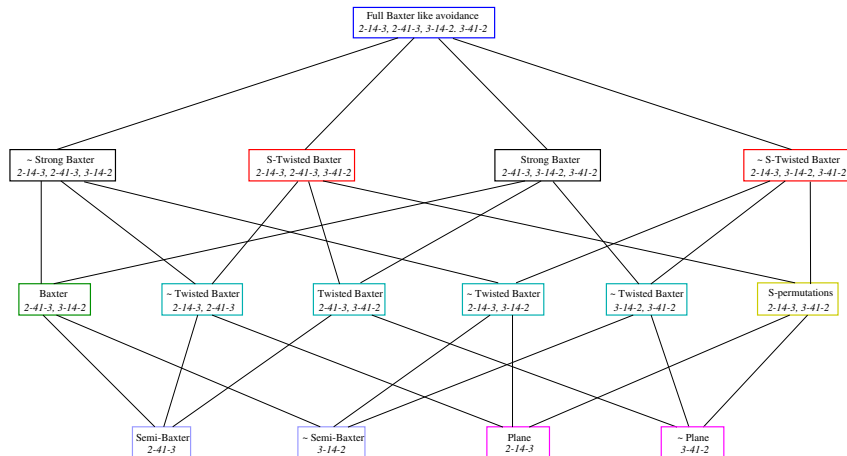
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Generating trees and ECO method

Generating trees

- A *generating tree* for (any combinatorial class) $\mathcal{C}$ is an infinite rooted tree, whose vertices are the objects of $\mathcal{C}$, each appearing exactly once in the tree, and such that objects of size n are at level n .
- The children of some object $c \in \mathcal{C}$ are obtained by adding an *atom* (i.e. a piece of object that makes its size increase by 1) to c .
- Since every object should appear only once in the tree, not all additions are possible.
- We should ensure the unique appearance property by considering only additions that follow some restricted rules.
- We will call the *growth* of $\mathcal{C}$ the process of adding atoms following these prescribed rules.

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Succession Rules

Succession rules

- A *succession rule* is a compact way of representing such a generating tree for a combinatorial class $\mathcal{C}$ without referring to its objects, but identifying them with *labels*.
- Made of one starting label (*label of the root*) and of *productions* encoding the way labels spread in the generating tree.
- Example:

$$\Omega_{Bax} = \left\{ \begin{array}{l} (1, 1) \\ (h, k) \rightsquigarrow (1, k+1) \dots (h, k+1) \\ (h+1, 1) \dots (h+1, k). \end{array} \right.$$

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- The n th term of the sequence is the total number of labels (counted with repetition) at level n in the generating tree.

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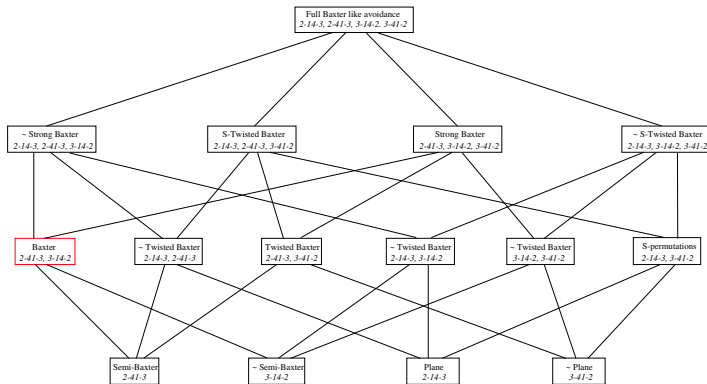
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Example: Baxter permutations

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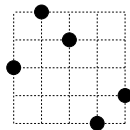
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Generating trees for Baxter permutations

Growth on the right of a permutation

For any π of size n and $a \in \{1, \dots, n+1\}$, $\pi \cdot a$ is the permutation π' of size $n+1$ such that $\pi'_{n+1} = a$, and

$$\pi'_i = \begin{cases} \pi_i & \text{if } \pi_i < a; \\ \pi_i + 1 & \text{if } \pi_i \geq a \end{cases}$$



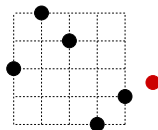
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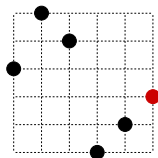
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Active sites

- The *active sites* of a Baxter permutation π are the points a such that $\pi \cdot a$ is a Baxter permutation.
- They are the points either immediately above a RTL maximum or immediately below a RTL minimum.
- The label of π is given by (h, k) where:
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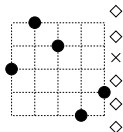
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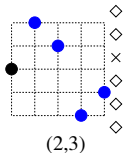
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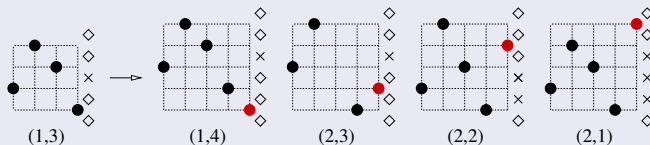
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- They are the points either immediately above a RTL maximum or immediately below a RTL minimum.
- The label of π is given by (h, k) where:
 - h = number of RTL minima.
 - k = number of RTL maxima.



Generating tree for Baxter permutations

Growth of a Baxter permutation



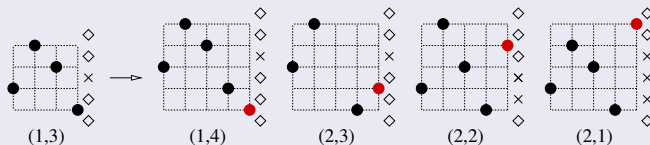
Succession rule for Baxter permutations

The growth of Baxter permutations can be described by

$$\Omega_{Bax} = \left\{ \begin{array}{l} (1, 1) \\ (h, k) \rightsquigarrow (1, k+1) \dots (h, k+1) \\ (h+1, 1) \dots (h+1, k). \end{array} \right.$$

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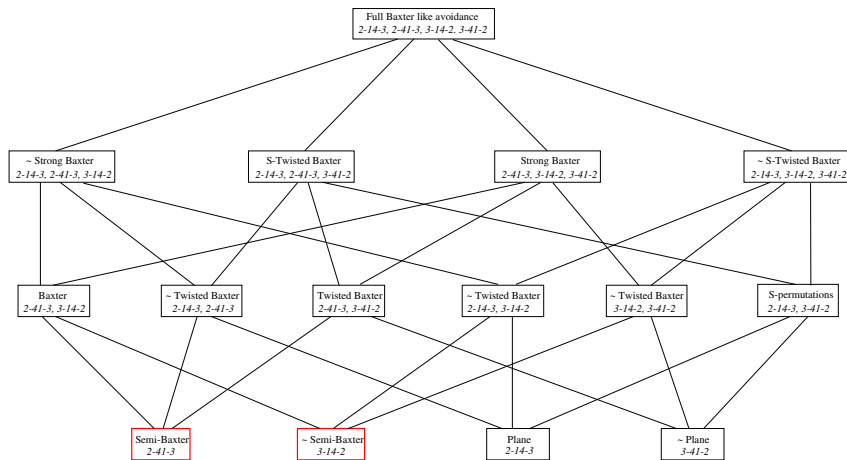


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Known facts

- *Semi-Baxter permutations* = $Av(2-41-3)$.
- Studied by Bouvel, Guerrini, Rechnitzer, R. (2016);
- Up to symmetry permutations avoiding 3-14-2 give the same sequence.
- Sequence of *semi-Baxter numbers* (sb_n) (A117106 in the OEIS).
- Begins 1, 2, 6, 23, 104, 530, 2958, 17734, ...
- Also studied by Pudwell as permutations avoiding the barred pattern $25\bar{3}14$.
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Generating tree for semi-Baxter permutations

Non-empty descents (ascents)

Let π be a permutation of length $n > 1$ and $1 < i \leq n$:

- π_i is a **non-empty descent** if it determines an occurrence of a pattern $2 - 31$;
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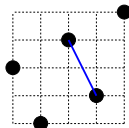
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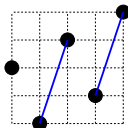


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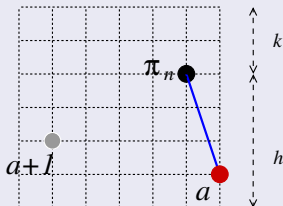
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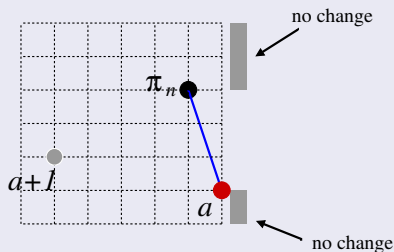
- $\pi \cdot a$ ends with a non-empty descent;
- all sites of π in the range $(a + 1, \pi_n + 1]$ become non-active in $\pi \cdot a$
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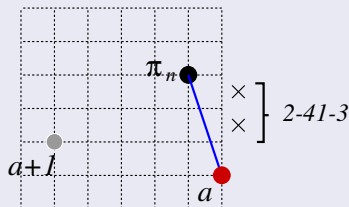
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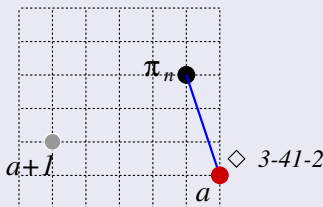
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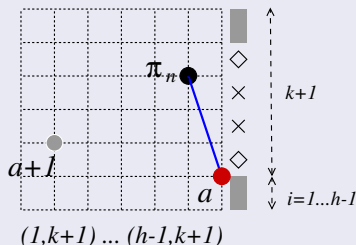
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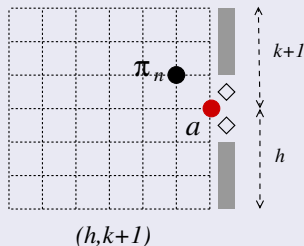
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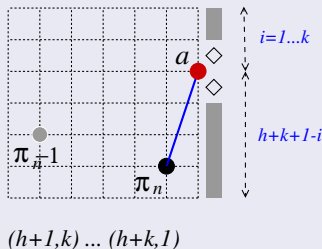
$\pi \cdot a$ ends with an empty descent: $\pi \cdot a$ has one active site more than π .



Generating tree for Semi-Baxter permutations

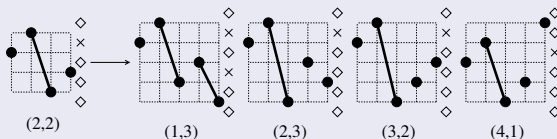
Case $a > \pi_n$

- $\pi \cdot a$ ends with an ascent;
- the active sites are not modified;
- the active site a is split in two active sites of $\pi \cdot a$



Generating tree for semi-Baxter permutations

Growth of a Semi-Baxter permutation



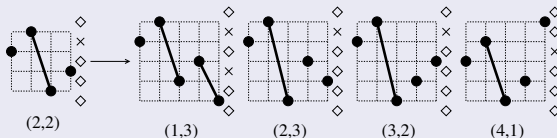
Succession rule for Semi-Baxter permutation

Described by the following succession rule:

$$\Omega_{semi} = \left\{ \begin{array}{l} (1, 1) \\ (h, k) \rightsquigarrow (1, k+1) \dots (h, k+1) \\ \quad (h+k, 1) \dots (h+1, k). \end{array} \right.$$

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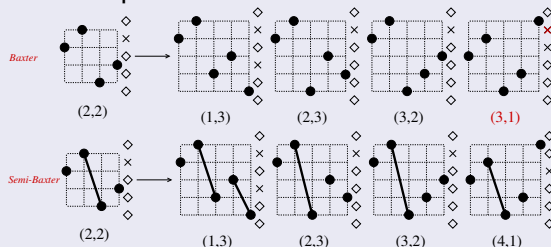
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Comparison of generating trees

Recursive growths for Baxter/Semi-Baxter permutations

- Every active site of a Baxter permutation is active for a semi-Baxter permutation;
- Every node in the Baxter generating tree can be mapped onto a node in the semi-Baxter generating tree;
- The growth of semi-Baxter permutations **extends** the growth of Baxter permutations.



Comparison of generating trees

Comparison of succession rules

We say that Ω_A *generalizes* Ω_B if

- 1 there is a comparison relation $\preceq$ between the labels of Ω_B and those of Ω_A ;
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The succession rule for semi-Baxter generalizes the one for Baxter permutations.

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- Consider the case where the growth for $\mathcal{B} \subseteq \mathcal{A}$ corresponding to $\Omega_{\mathcal{B}}$ specializes the growth for $\mathcal{A}$ corresponding to $\Omega_{\mathcal{A}}$, in the sense that for any object b of $\mathcal{B}$ (which is of course also an object of $\mathcal{A}$), the set of *active sites* of b as an object of $\mathcal{B}$ is a subset of the set of active sites of b as an object of $\mathcal{A}$.
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Properties of Semi-Baxter numbers

A recurrence relation

The numbers sb_n are recursively characterized by $sb_0 = 0$, $sb_1 = 1$ and for $n \geq 2$

$$sb_n = \frac{11n^2 + 11n - 6}{(n+4)(n+3)} sb_{n-1} + \frac{(n-3)(n-2)}{(n+4)(n+3)} sb_{n-2} \quad (1)$$

A formula for semi-Baxter numbers

From (1) a formula for semi-Baxter numbers can be obtained, previously conjectured by Bevan ($n \geq 2$):

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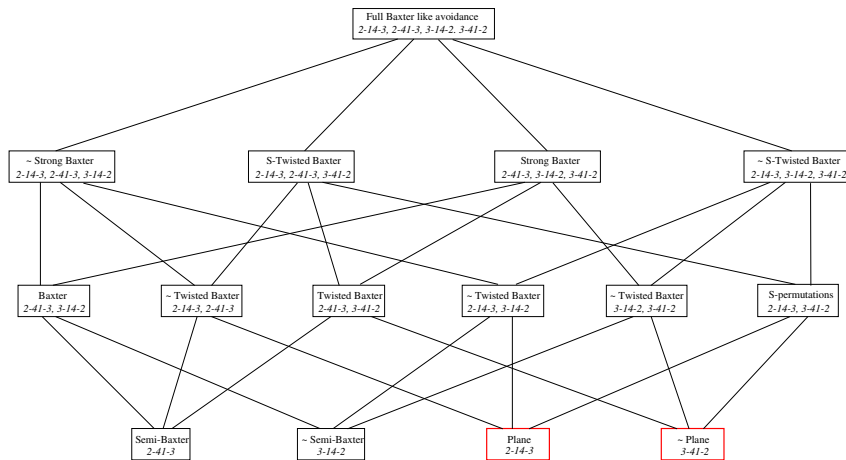
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Plane permutations = $Av(2-14-3)$



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The Hasse graph of a permutation

- **Hasse graph** of a permutation π of size n : the oriented graph on the vertex set $\{1, \dots, n\}$, which includes an edge from i to j (for $i < j$) if and only if $\pi(i) < \pi(j)$ and there is no k such that $i < k < j$ and $\pi(i) < \pi(k) < \pi(j)$, and with all edges pointing upward.
- Hasse graphs of 2413 and 2143:



2413



2143

Plane permutations = $Av(2-14-3)$

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- **Hasse graph** of a permutation π of size n : the oriented graph on the vertex set $\{1, \dots, n\}$, which includes an edge from i to j (for $i < j$) if and only if $\pi(i) < \pi(j)$ and there is no k such that $i < k < j$ and $\pi(i) < \pi(k) < \pi(j)$, and with all edges pointing upward.
- Hasse graphs of 2413 and 2143:



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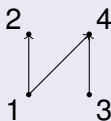


2143

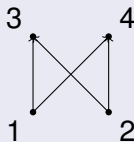
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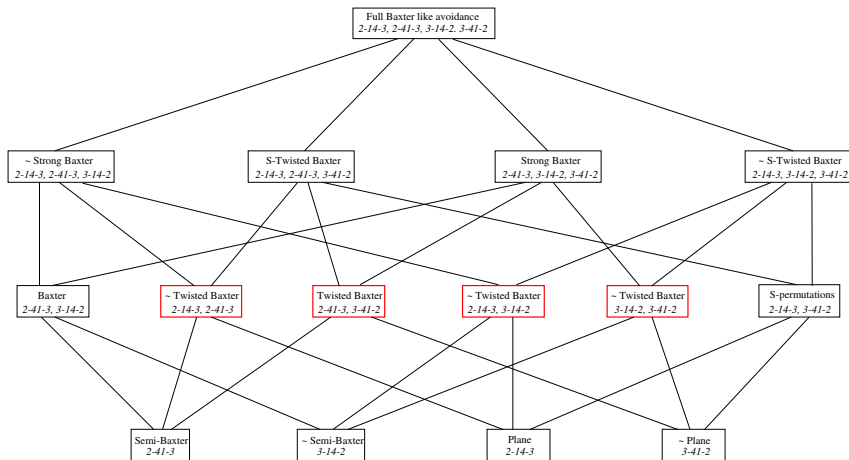
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$$\Omega_{TBax} = \left\{ \begin{array}{l} (1, 1) \\ (h, k) \rightsquigarrow (1, k) \dots (h-1, k) \\ \quad (h, k+1) \\ \quad (h+k, 1) \dots (h+1, k). \end{array} \right.$$

- This (asymmetric) generating tree is another specialization of the generating tree for semi-Baxter permutations.

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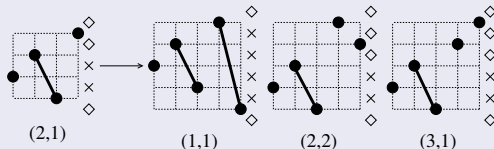
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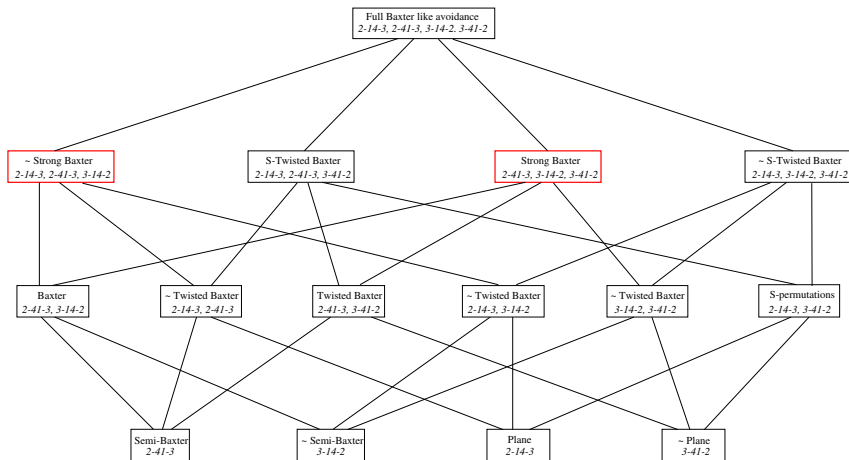
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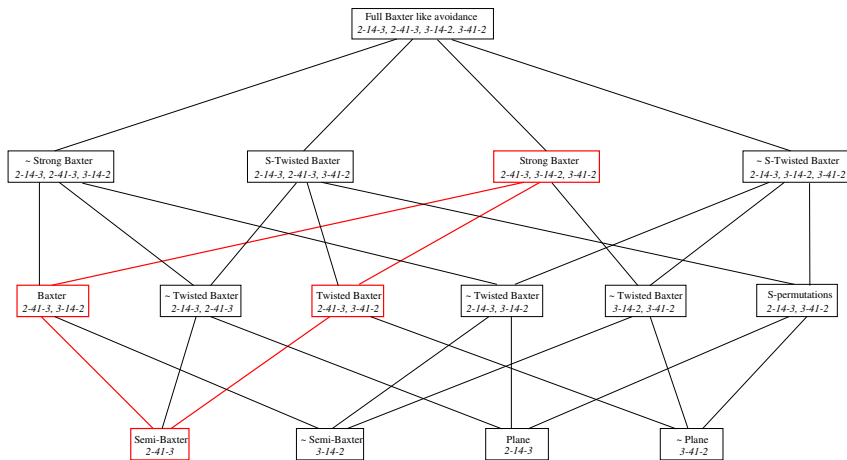
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- Studied by Bouvel, Guerrini, Rechnitzer, R., (2016);
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- Sequence begins 1, 2, 6, 21, 82, 346, 1547, 7236, 35090, ... (sequence A281784 in OEIS).
- It is proved that the generating function of strong-Baxter numbers is not D-finite, and that their growth constant is $\rho_W + 2 \approx 6.729031538$.
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- A recursive growth for strong Baxter permutations (Bouvel, Guerrini, Rechnitzer, R., 2016).
- To a permutation π we assign the label (h, k) , where h (resp. k) is the number of active sites smaller than or equal to (resp. greater than) π_n ;

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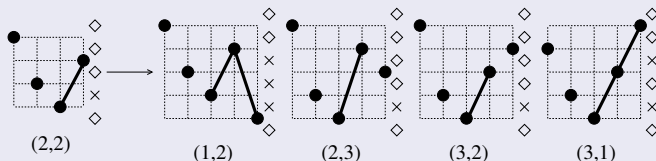
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Comparison of succession rules

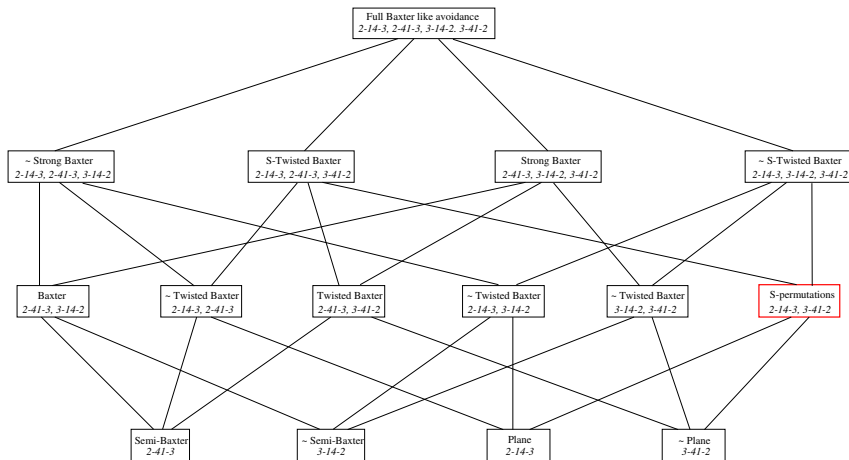
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Proposition. For all families above, the succession rule for a smaller family is a specialization of the one for a larger family.

$\Omega_{semi} :$	(h, k)	$\rightsquigarrow$	$(1, k+1)$	$\dots$	$(h-1, k+1)$	$(h, k+1)$	$(h+k, 1)$	$\dots$	$(h+1, k)$
$\Omega_{Bax} :$	(h, k)	$\rightsquigarrow$	$(1, k+1)$	$\dots$	$(h-1, k+1)$	$(h, k+1)$	$(h+1, 1)$	$\dots$	$(h+1, k)$
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S -permutations = (2-14-3, 3-41-2):



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Definitions and known facts

- *S-permutations*: Asinowski, Barequet, Bousquet-Mélou, Mansour, Pinter, (2013);
- Define an order relation S between the segments of floorplans.
- This leads to the notion of S -equivalent floorplans, and to construct the S -permutation $S(P)$ of a floorplan P .
- Prove that $S\text{-permutations} = \text{Av}(2\text{-}14\text{-}3, 3\text{-}41\text{-}2)$.

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Definitions and known facts

- The number of S -permutations of size n :

$$a_n = \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} (-1)^i \binom{n+1-i}{i} b_{n+1-i} \quad (2)$$

with b_n being the n th Baxter number

Asinowski, Barequet, Bousquet-Mélou, Mansour, Pinter, (2013)

- The first terms of the sequence a_n are:
1, 2, 6, 22, 88, 374, 1668, 7744, 37182, ... (A214358 in OEIS).

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- Obtained via a bijection between S -permutations of size n and Baxter permutations of size $n+1$ in which all maximal ascending F -blocks are increasing (i.e. order isomorphic to a permutation of the form $123 \dots m$), and all maximal descending F -blocks of size at least 3 are decreasing.

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Definitions and known facts

- The number of *S*-permutations of size n :

$$a_n = \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} (-1)^i \binom{n+1-i}{i} b_{n+1-i} \quad (4)$$

with b_n being the n th Baxter number

Asinowski, Barequet, Bousquet-Mélou, Mansour, Pinter, (2013)

- Even though the enumeration of the *S*-permutations has been solved, the proof (which does not use generating trees) does not allow to include the *S*-permutations in the comparison of succession rules shown above.

Generating tree for S -permutations

Generating tree for S -permutations

- We can let S -permutations grow “on the right”.
- Any label (h, k) records the number h (resp. k) of active sites above (below) the rightmost point of the corresponding permutation.
- The growth restricts that of plane permutations.
- Generating tree with three-color labels. The color of label (h, k) of permutation π is determined by π_n :



green label



blue label



red label

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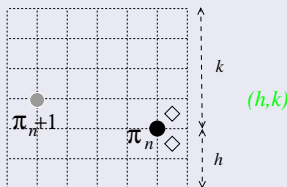


red label

Generating tree for S-permutations (2-14-3, 3-41-2): green labels

Case $a > \pi_n + 1$

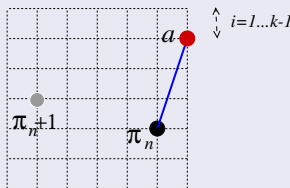
- $\pi \cdot a$ ends with a non-empty ascent ;
- all sites in $(\pi_n + 1, a]$ become non-active in $\pi \cdot a$;
- all other sites remain active in $\pi \cdot a$ if they were active in π .



Generating tree for S-permutations (2-14-3, 3-41-2): green labels

Case $a > \pi_n + 1$

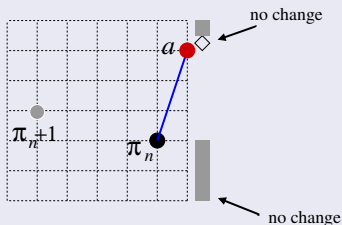
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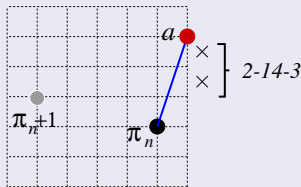
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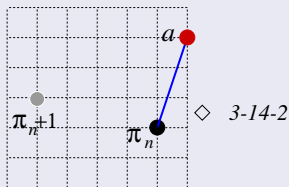
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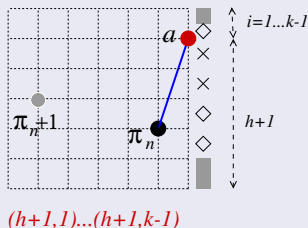
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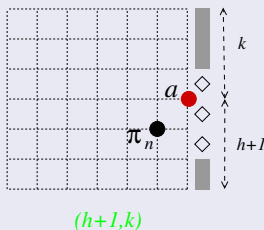
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Generating tree for S-permutations (2-14-3, 3-41-2): green labels

Case $a = \pi_n + 1$

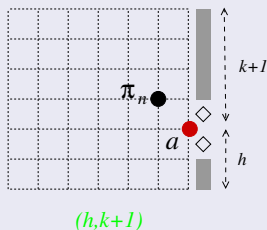
- $\pi \cdot a$ ends with an empty ascent;
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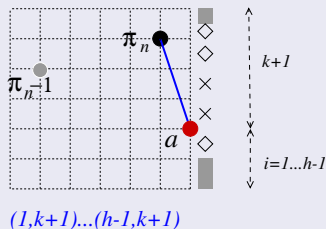
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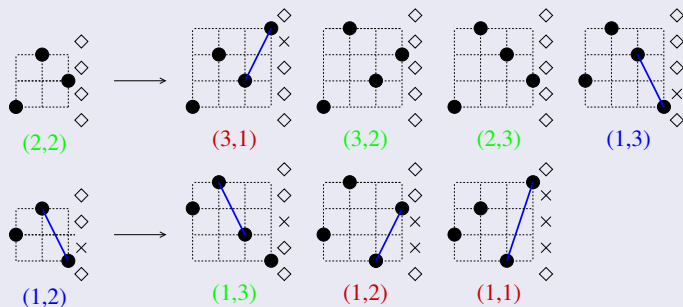
- $\pi \cdot a$ ends with a non-empty descent (symmetric to the case of a non-empty ascent);
- all sites in $(a, \pi_n - 1]$ become non-active in $\pi \cdot a$;
- all other sites remain active in $\pi \cdot a$ if they were active in π .



Generating tree for S-permutations: red and blue labels

Case of red and blue labels

- the situation is analogous to the case of green labels;
- the only difference is in the point immediately above (or below) π_n



Succession rule for S-permutations

Succession rule for S-permutations

The growth of S-permutations can be described by the following succession rule

$$\Omega_S = \left\{ \begin{array}{l} (1, 1) \\ (h, k) \rightsquigarrow (h+1, 1) \dots (h+1, k-1) (h+1, k) \\ \qquad \qquad \qquad (h, k+1) (h-1, k+1) \dots (1, k+1) \\ \\ (h, k) \rightsquigarrow (h+1, 1) \dots (h+1, k-1) (h+1, k) \\ \qquad \qquad \qquad (h, k) (h-1, k) \dots (1, k) \\ \\ (h, k) \rightsquigarrow (h, 1) \dots (h, k-1) (h, k) \\ \qquad \qquad \qquad (h, k+1) (h-1, k+1) \dots (1, k+1). \end{array} \right.$$

Enumeration of S-permutations

Generating function of S-permutations

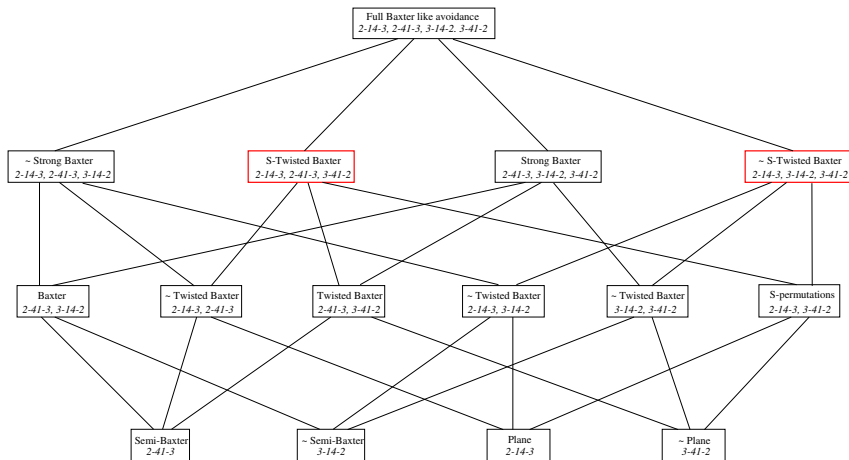
The above three-color generating tree yields a system of functional equations:

$$\left\{ \begin{array}{l} F(y, z) = xyz + x(y+z)F(y, z) + xyR(y, z) + xzB(y, z) \\ R(y, z) = \frac{yxz}{1-z} (R(y, 1) + F(y, 1)) - \frac{yx}{1-z} (R(y, z) + F(y, z)) \\ \quad + \frac{xz}{1-z} (B(y, 1) - B(y, z)) \\ B(y, z) = \frac{yxz}{1-y} (B(1, z) + F(1, z)) - \frac{zx}{1-y} (B(y, z) + F(y, z)) \\ \quad + \frac{xy}{1-y} (R(1, z) - R(y, z)) . \end{array} \right.$$

where $F(y, z)$ ($R(y, z)$, $B(y, z)$) is the generating function of green (red, blue) labels. We are working on re-deriving the enumeration of S-permutations from these equations.

Twisted Baxter S-permutations

$= (2-14-3, 3-14-2, 3-41-2)$



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Twisted Baxter S-permutations

- Permutations avoiding $(2-14-3, 3-14-2, 3-41-2)$ (and the equivalent family $(2-41-3, 2-14-3, 3-41-2)$) have not been studied in the literature as far as we know;
- Obtained as the intersection of S-permutations and twisted Baxter permutations;
- The first terms of the sequence are:
 $1, 2, 6, 21, 80, 322, 1354, 5901, 26494, 121960, 573458, \dots$
- Generating tree describing its growth which specializes Ω_S and requires three-color labels as well (we use green, red, and blue labels).

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Succession rule

Succession rule for twisted Baxter S-permutations

Any colored label (h, k) records the number h (resp. k) of active sites below (above) the rightmost point of the corresponding permutation

$$\Omega_{Stwist} = \left\{ \begin{array}{l} (1, 1) \\ (h, k) \rightsquigarrow (h, 1) \dots (h, k-1) (h+1, k) \\ (h, k+1) (h-1, k) \dots (1, k) \\ \\ (h, k) \rightsquigarrow (h, 1) \dots (h, k-1) (h+1, k) \\ (h, k) (h-1, k) \dots (1, k) \\ \\ (h, k) \rightsquigarrow (h, 1) \dots (h, k-1) \\ (h, k+1) (h-1, k) \dots (1, k). \end{array} \right.$$

Specializing Ω_S .

Enumeration of twisted S-permutations

Functional equation

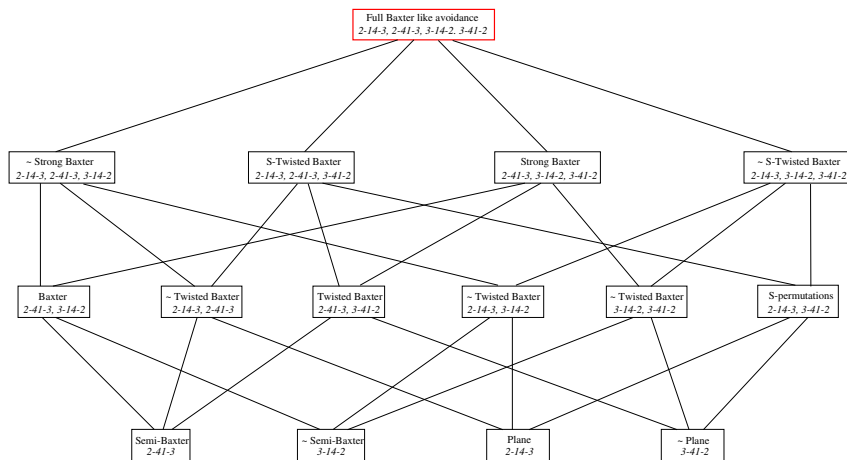
The above three-color generating tree yields the following system of functional equations, where $F(y, z)$ ($R(y, z)$, $B(y, z)$) is the generating function of black (red, blue) labels.

$$\begin{cases} F(y, z) &= xyz + x(y+z)F(y, z) + xyR(y, z) + xzB(y, z) \\ R(y, z) &= \frac{xz}{1-z} (R(y, 1) + F(y, 1) + B(y, 1)) - \frac{x}{1-z} (R(y, z) + F(y, z) + zB(y, z)) \\ B(y, z) &= \frac{yxz}{1-y} (B(1, z) + F(1, z)) - \frac{zx}{1-y} (B(y, z) + F(y, z)) + \frac{xy}{1-y} (R(1, z) - R(y, z)) . \end{cases}$$

We are working on these equations to obtain the enumeration of twisted Baxter S-permutations or the nature of their generating function.

Full Baxter-like avoiding permutations

= (2-41-3, 3-14-2, 2-14-3, 3-41-2)



Full Baxter-like avoiding permutations

$$= (2-41-3, 3-14-2, 2-14-3, 3-41-2)$$

F-permutations

- Family not known in the literature (briefly, *F-permutations*).
- It is the smallest family and comprises permutations which belong to all the previous families.
- The first terms of the sequence are:
1, 2, 6, 20, 72, 274, 1088, 4470, 18884, 81652, 360054, ...
- The growth restricts that of twisted Baxter S-permutations, and the succession rule describing its growth specializes Ω_{Stwis} , with three-color labels:



green label



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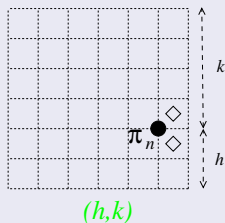


red label

Generating tree for F-permutations: green labels

Case $a > \pi_n + 1$

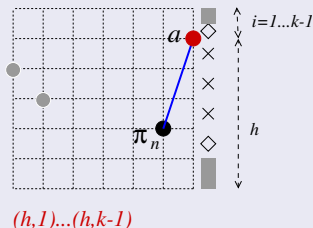
- $\pi \cdot a$ ends with a non-empty ascent ;
- all sites in $[\pi_{n+1}, a]$ become non-active in $\pi \cdot a$;
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Generating tree for F-permutations: green labels

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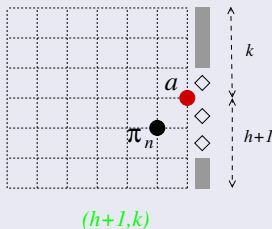
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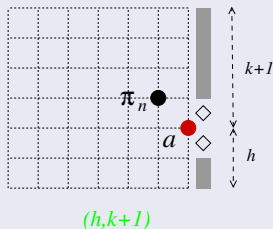
- $\pi \cdot a$ ends with an empty ascent;
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Generating tree for S-permutations: green labels

Case $a = \pi_n$

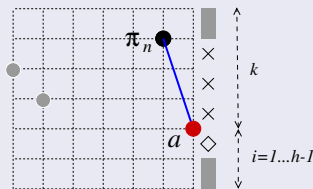
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Generating tree for S-permutations: green labels

Case $a < \pi_n$

- $\pi \cdot a$ ends with a non-empty descent (symmetric to the case of a non-empty ascent);
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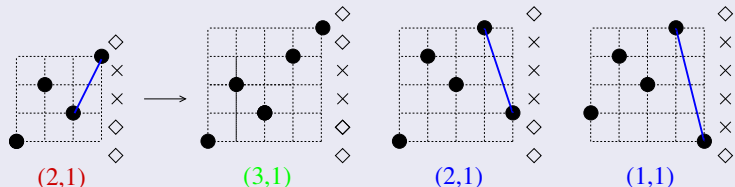


$(1, k) \dots (h-1, k)$

Generating tree for F-permutations: red and blue labels

Case of red and blue labels

- the situation is analogous to the case of green labels;
- the only difference is in the point immediately above (or below) π_n



Succession rule for F-permutations

Succession rule for F-permutations

Any colored label (h, k) records the number h (resp. k) of active sites below (above) the rightmost point of the corresponding permutation

$$\Omega_{Full} = \left\{ \begin{array}{l} (1, 1) \\ (h, k) \rightsquigarrow (h, 1) \dots (h, k-1) (h+1, k) \\ (h, k+1) (h-1, k) \dots (1, k) \\ \\ (h, k) \rightsquigarrow (h, 1) \dots (h, k-1) (h+1, k) \\ (h, k) (h-1, k) \dots (1, k) \\ \\ (h, k) \rightsquigarrow (h, 1) \dots (h, k-1) (h, k) \\ (h, k+1) (h-1, k) \dots (1, k). \end{array} \right.$$

Specializing Ω_{Stwis} .

Enumeration of F-permutations

Functional equation

The above three-color generating tree yields the following system of functional equations, where $F(y, z)$ ($R(y, z)$, $B(y, z)$) is the generating function of green (red, blue) labels.

$$\begin{cases} F(y, z) &= xyz + x(y+z)F(y, z) + xyR(y, z) + xzB(y, z) \\ R(y, z) &= \frac{xz}{1-z} (R(y, 1) + F(y, 1) + B(y, 1)) - \frac{x}{1-z} (R(y, z) + F(y, z) + zB(y, z)) \\ B(y, z) &= \frac{yx}{1-y} (B(1, z) + F(1, z) + R(1, z)) - \frac{x}{1-y} (B(y, z) + F(y, z) + yR(y, z)) . \end{cases}$$

We are working on the enumeration of F-permutations from these equations and in discovering the nature of their generating function.

Recursive growth for permutations avoiding Baxter-like patterns

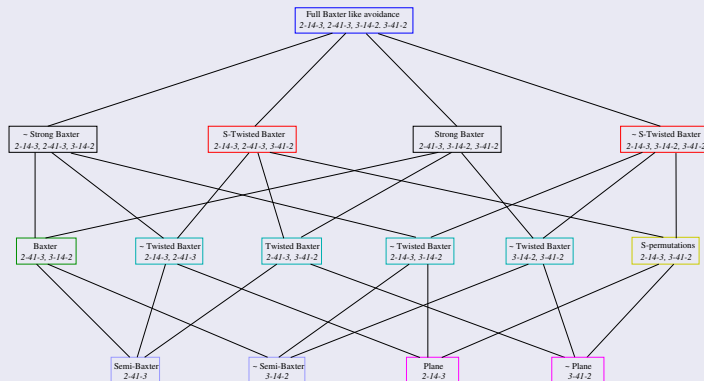
Recursive growth

For any family, the recursive growth “on the right” is a restriction of the growth for the larger families.

Recursive growth for permutations avoiding Baxter-like patterns

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Succession rules for permutations avoiding Baxter-like patterns

Succession rules

Families avoiding Baxter-like patterns can be partitioned in two groups:

- (1.) growing according to succession rules specializing Ω_{semi} ;
- (2.) growing according to succession rules specializing Ω_S (three labels).

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