

Packing patterns in restricted permutations

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Notation

- $\nu(\rho, \pi)$ is the number of occurrences of ρ in π .
- $d(\rho) = \lim_{n \rightarrow \infty} \frac{\max_{\pi \in \mathcal{S}_n} \nu(\rho, \pi)}{\binom{n}{|\rho|}}$ (packing density)

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- $d(12 \cdots m) = 1$ (Pack $12 \cdots m$ into $12 \cdots n$.)

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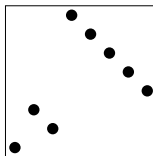
- $d(12 \cdots m) = 1$ (Pack $12 \cdots m$ into $12 \cdots n$.)
- For all $\rho \in \mathcal{S}_m$, $d(\rho)$ exists.

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Known:

- $d(12 \cdots m) = 1$ (Pack $12 \cdots m$ into $12 \cdots n$.)
- For all $\rho \in \mathcal{S}_m$, $d(\rho)$ exists.
- If ρ is layered, then $\max_{\pi \in \mathcal{S}_n} \nu(\rho, \pi)$ is achieved by a layered π .
- $d(132) = 2\sqrt{3} - 3 \approx 0.464$



Notation

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Previous work:

$$d(\rho) = \lim_{n \rightarrow \infty} \frac{\max_{\pi \in \mathcal{S}_n} \nu(\rho, \pi)}{\binom{n}{|\rho|}}$$

Notation

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Previous work:

$$d(\rho) = \lim_{n \rightarrow \infty} \frac{\max_{\pi \in \mathcal{S}_n} \nu(\rho, \pi)}{\binom{n}{|\rho|}}$$

In this talk:

$$d_{\sigma}(\rho) = \lim_{n \rightarrow \infty} \frac{\max_{\pi \in \mathcal{S}_n(\sigma)} \nu(\rho, \pi)}{\binom{n}{|\rho|}} \qquad d_A(\rho) = \lim_{n \rightarrow \infty} \frac{\max_{\pi \in A_n} \nu(\rho, \pi)}{\binom{n}{|\rho|}}$$

A_n is the set of *alternating permutations*,
i.e. those that avoid consecutive 123 patterns and consecutive 321 patterns.

Packing patterns of length 3

Recall:
$$d_\sigma(\rho) = \lim_{n \rightarrow \infty} \frac{\max_{\pi \in \mathcal{S}_n(\sigma)} \nu(\rho, \pi)}{\binom{n}{|\rho|}}$$

$\rho \backslash \sigma$	123	132	213	231	312	321	-
123							1
132							$2\sqrt{3} - 3$

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123	0	1	1	1	1	1	1
132		0					$2\sqrt{3} - 3$

- $I_n = 12 \cdots n$ avoids $\sigma \in \mathcal{S}_3 \setminus \{123\}$.

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123	0	1	1	1	1	1	1
132		0		$2\sqrt{3} - 3$	$2\sqrt{3} - 3$		$2\sqrt{3} - 3$

- $I_n = 12 \cdots n$ avoids $\sigma \in \mathcal{S}_3 \setminus \{123\}$.
- Layered permutations avoid 231 and 312.

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$\rho \backslash \sigma$	123	132	213	231	312	321	-
123	0	1	1	1	1	1	1
132	?	0	?	$2\sqrt{3} - 3$	$2\sqrt{3} - 3$	?	$2\sqrt{3} - 3$

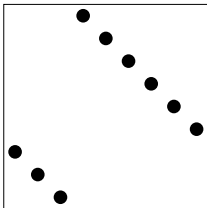
- $I_n = 12 \cdots n$ avoids $\sigma \in \mathcal{S}_3 \setminus \{123\}$.
- Layered permutations avoid 231 and 312.
- New: $d_{123}(132)$, $d_{213}(132)$, and $d_{321}(132)$

Packing 132

...and avoiding 123

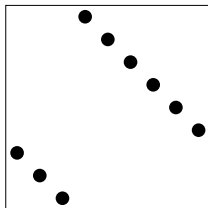
Packing 132

...and avoiding 123



Packing 132

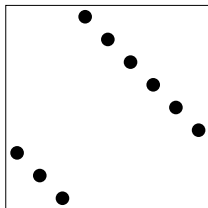
...and avoiding 123



- $J_i \oplus J_{n-i}$ has $i \binom{n-i}{2}$ copies of 132. ($J_n = n \cdots 21$)

Packing 132

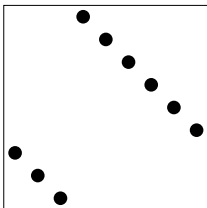
...and avoiding 123



- $J_i \oplus J_{n-i}$ has $i \binom{n-i}{2}$ copies of 132. ($J_n = n \cdots 21$)
- Maximized when $i = \lfloor \frac{n}{3} \rfloor$.

Packing 132

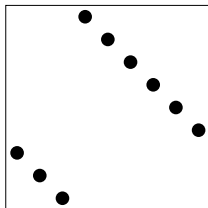
...and avoiding 123



- $J_i \oplus J_{n-i}$ has $i \binom{n-i}{2}$ copies of 132. ($J_n = n \cdots 21$)
- Maximized when $i = \lfloor \frac{n}{3} \rfloor$.
- Implies $d_{123}(132) = \frac{4}{9}$.

Packing 132

...and avoiding 123

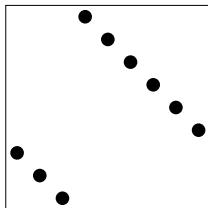


...and avoiding 213

- $J_i \oplus J_{n-i}$ has $i \binom{n-i}{2}$ copies of 132. ($J_n = n \cdots 21$)
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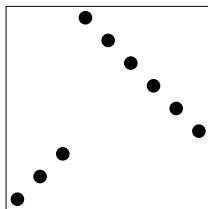
Packing 132

...and avoiding 123



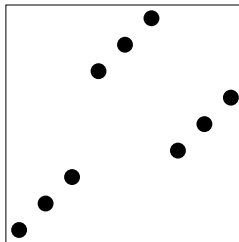
- $J_i \oplus J_{n-i}$ has $i \binom{n-i}{2}$ copies of 132. ($J_n = n \cdots 21$)
- Maximized when $i = \lfloor \frac{n}{3} \rfloor$.
- Implies $d_{123}(132) = \frac{4}{9}$.

...and avoiding 213



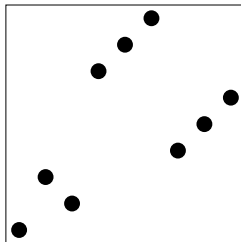
- $l_i \oplus J_{n-i}$ has $i \binom{n-i}{2}$ copies of 132.
- Maximized when $i = \lfloor \frac{n}{3} \rfloor$.
- Implies $d_{213}(132) = \frac{4}{9}$.

Packing 132 and Avoiding 321



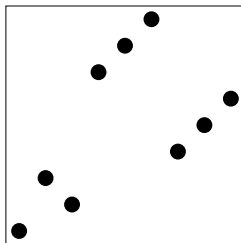
- $I_a \oplus (I_b \ominus I_c)$ has $a \cdot b \cdot c$ copies of 132.

Packing 132 and Avoiding 321



- $I_a \oplus (I_b \ominus I_c)$ has $a \cdot b \cdot c$ copies of 132.
- Replace initial I_a with a 132-optimizer of length a to get more copies.

Packing 132 and Avoiding 321



- $I_a \oplus (I_b \ominus I_c)$ has $a \cdot b \cdot c$ copies of 132.
- Replace initial I_a with a 132-optimizer of length a to get more copies.
- Optimized when $a = \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) n$, $b = c = \left(\frac{3}{4} - \frac{\sqrt{3}}{4}\right) n$.
- Implies $d_{321}(132) = \sqrt{3} - \frac{3}{2}$.

Recap:

$$d_{\sigma}(\rho) = \lim_{n \rightarrow \infty} \frac{\max_{\pi \in S_n(\sigma)} \nu(\rho, \pi)}{\binom{n}{|\rho|}}$$

$\rho \backslash \sigma$	123	132	213	231	312	321	-
123	0	1	1	1	1	1	1
132	$\frac{4}{9}$	0	$\frac{4}{9}$	$2\sqrt{3} - 3$	$2\sqrt{3} - 3$	$\sqrt{3} - \frac{3}{2}$	$2\sqrt{3} - 3$

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123	0	1	1	1	1	1	1
132	$\frac{4}{9}$	0	$\frac{4}{9}$	$2\sqrt{3} - 3$	$2\sqrt{3} - 3$	$\sqrt{3} - \frac{3}{2}$	$2\sqrt{3} - 3$

Or approximately...

$\rho \backslash \sigma$	123	132	213	231	312	321	-
123	0	1	1	1	1	1	1
132	0.444	0	0.444	0.464	0.464	0.232	0.464

Packing patterns of length 4

Recall:
$$d_\sigma(\rho) = \lim_{n \rightarrow \infty} \frac{\max_{\pi \in \mathcal{S}_n(\sigma)} \nu(\rho, \pi)}{\binom{n}{|\rho|}}$$

$\rho \backslash \sigma$	123	132	213	231	312	321	-
1234							1
1432							α
2143							$\frac{3}{8}$
1243							$\frac{3}{8}$
1324							≈ 0.244
1342							≈ 0.19658
2413							≈ 0.10474

α is the real root of $x^3 - 12x^2 + 156x - 64$ (≈ 0.42357)

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1243	0	0		$\frac{3}{8}$	$\frac{3}{8}$		$\frac{3}{8}$
1324	0	0	0	β	β		$\approx 0.244 (\beta)$
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1243	0	0	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$		$\frac{3}{8}$
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1432	$\frac{27}{64}$	0	$\frac{27}{64}$	α	α	0	α
2143	$\frac{3}{8}$	0	0	$\frac{3}{8}$	$\frac{3}{8}$	$\geq \frac{3}{32}$	$\frac{3}{8}$
1243	0	0	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\geq \frac{3}{16}$	$\frac{3}{8}$
1324	0	0	0	β	β	$\geq \gamma$	≈ 0.244 (β)
1342	0	0	$\geq \frac{3}{16}$	0	$\geq \frac{3}{16}$	$\geq \delta$	≈ 0.19658
2413	$\geq \frac{3}{32}$	0	0	0	0	$\geq \frac{3}{32}$	≈ 0.10474

α is the real root of $x^3 - 12x^2 + 156x - 64$ (≈ 0.42357)

$$\frac{3}{16} = 0.1875, \quad \frac{3}{32} = 0.09375, \quad \frac{27}{64} = 0.421875$$

$$\gamma \approx 0.09450, \quad \delta \approx 0.18825$$

Alternating Permutations

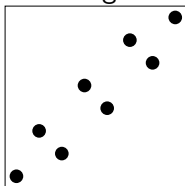
A_n is the set of permutations of length n avoiding 123 and 321 consecutively.

1324	1423	2314	2413	3412
4231	4132	3241	3142	2143

Goal: Find $d_A(\rho) = \lim_{n \rightarrow \infty} \frac{\max_{\pi \in A_n} \nu(\rho, \pi)}{\binom{n}{|\rho|}}$

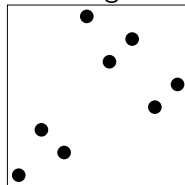
Alternating packing densities

Packing 123



- $1 \oplus 21 \oplus \cdots \oplus 21 (\oplus 1)$
- Implies $d_A(123) = 1$.

Packing 132



- Use same ratios for “alternating layers” as 132-optimizer in $\mathcal{S}_n$.
- Implies $d_A(132) = 2\sqrt{3} - 3$.

Counting Sequences

Let $a_{123}(n)$ be the number of copies of 123 in $1 \oplus 21 \oplus \cdots \oplus 21 (\oplus 1)$.

$$a_{123}(n) = \begin{cases} 2\left(\frac{n}{2} - 1\right) + 8\binom{\frac{n}{2}-1}{2} + 8\binom{\frac{n}{2}-1}{3} & n \text{ even} \\ 4\binom{\frac{n-1}{2}}{2} + 8\binom{\frac{n-1}{2}}{3} & n \text{ odd} \end{cases}$$

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2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, ...

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A099956	Atomic numbers of the alkaline earth metals.	9
	4, 12, 20, 38, 56, 88 (list ; graph ; refs ; listen ; history ; text ; internal format)	
OFFSET	1,1	
LINKS	Table of n, a(n) for n=1..6.	
EXAMPLE	12 is the atomic number of magnesium.	
CROSSREFS	Cf. A099955 , alkali metals; A101648 , metalloids; A101647 , nonmetals (except halogens and noble gases); A097478 , halogens; A018227 , noble gases; A101649 , poor metals.	
	Sequence in context: A057317 A008068 A008183 * A301066 A008092 A316299	
	Adjacent sequences: A099953 A099954 A099955 * A099957 A099958 A099959	
KEYWORD	nonn,fini,full	
AUTHOR	Parthasarathy Nambi , Nov 12 2004	
STATUS	approved	

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2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, ...

[A168380](#)

Row sums of [A168281](#).

+20
14

2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, 1140, 1340, 1540, 1782, 2024, 2312, 2600, 2938, 3276, 3668, 4060, 4510, 4960, 5472, 5984, 6562, 7140, 7788, 8436, 9158, 9880, 10680, 11480, 12362, 13244, 14212, 15180, 16238, 17296, 18448, 19600, 20850, 22100 ([list](#); [graph](#); [refs](#); [listen](#); [history](#); [text](#); [internal format](#))

OFFSET

1,1

COMMENTS

The atomic numbers of the augmented alkaline earth group in Charles Janet's spiral periodic table are 0 and the first eight terms of this sequence (see Stewart reference). - [Alonso del Arte](#), May 13 2011

LINKS

Vincenzo Librandi, [Table of n, a\(n\) for n = 1..10000](#)
Stewart, Philip, [Charles Janet: unrecognized genius of the Periodic System](#).
Foundations of Chemistry (2010), p. 9.
[Index entries for linear recurrences with constant coefficients](#), signature (2,1,-4,1,2,-1).

FORMULA

$a(n) = 2 \cdot A005993(n-1)$.
 $a(n) = (n+1) \cdot (3 + 2 \cdot n^2 + 4 \cdot n - 3 \cdot (-1)^n) / 12$.
 $a(n+1) - a(n) = A093907(n) = A137583(n+1)$.
 $a(2n+1) = A035597(n+1)$ $a(2n) = A002492(n)$.

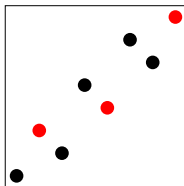
Alkaline Earth Metals (Group 2)

Group Period	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 H																	2 He
2	3 Li	4 Be											5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg											13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	57 La	* 72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	89 Ac	* 104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Nh	114 Fl	115 Mc	116 Lv	117 Ts	118 Og
				* 58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu	
				* 90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr	

Notation for copies of 123 in $1 \oplus 21 \oplus \cdots \oplus 21 (\oplus 1)$

Observation: copies of 123 come in pairs.

$$1 \oplus 21 \oplus \cdots \oplus 21 \oplus 1$$

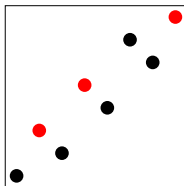


Given xyz embedding of 123 where y is even, $x(y+1)z$ is also a 123.
We will assign a tuple of integers to each such pair.

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xyz corresponds to the tuple (n, ℓ, m) where...

- $|m|$ is the layer where x is found (count layers starting with 0).
- m is negative if we use the smaller entry in the layer as x , positive if we use the larger entry.
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- $n + \ell = z - 3$.

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Example: $1 \oplus 21 \oplus 21 \oplus 21 = 1325476$

copies	tuple	copies	tuple	copies	tuple
124,134	(1,0,0)	146,156	(2,1,0)	147,157	(3,1,0)
125,135	(2,0,0)	246,256	(2,1,-1)	247,257	(3,1,-1)
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A little chemistry...

- *Quantum numbers* describe trajectories of electrons.

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- ▶ Two possible spin numbers for each choice of (n, ℓ, m)

Alkaline Earth Metals (Group 2)

Electron Configurations in the Periodic Table

1 H 1s																	2 He 1s																												
3 Li 2s	4 Be																	5 B	6 C	7 N	8 O	9 F	10 Ne 2p																						
11 Na 3s	12 Mg																	13 Al	14 Si	15 P	16 S	17 Cl	18 Ar 3p																						
19 K 4s	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr 4p																												
37 Rb 5s	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe 5p																												
55 Cs 6s	56 Ba	57 La	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn 6p																												
87 Fr 7s	88 Ra	89 Ac	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110	111	112	113	114																																
		<table> <tr> <td>58 Ce</td><td>59 Pr</td><td>60 Nd</td><td>61 Pm</td><td>62 Sm</td><td>63 Eu</td><td>64 Gd</td><td>65 Tb</td><td>66 Dy</td><td>67 Ho</td><td>68 Er</td><td>69 Tm</td><td>70 Yb</td><td>71 Lu</td> </tr> <tr> <td>90 Th</td><td>91 Pa</td><td>92 U</td><td>93 Np</td><td>94 Pu</td><td>95 Am</td><td>96 Cm</td><td>97 Bk</td><td>98 Cf</td><td>99 Es</td><td>100 Fm</td><td>101 Md</td><td>102 No</td><td>103 Lr</td> </tr> </table>																58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr
58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu																																
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by: Sarah Faldt

Subshell is (n, ℓ) with ℓ given by **s** ($\ell = 0$), **p** ($\ell = 1$), **d** ($\ell = 2$), **f** ($\ell = 3$).
 e.g. Calcium has subshells of the form **1s**, **2s**, **2p**, **3s**, **3p**, **4s**.

123s to electrons

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Subshell: Know n and ℓ . Need all tuples (n, ℓ, m) where $-\ell \leq m \leq \ell$

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We saw the copies of 123 in $1 \oplus 21 \oplus 21 \oplus 21 = 1325476$ are:

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Future Directions

- Bijections between pattern embeddings and other combinatorial objects.
- Determine $d_\sigma(\rho)$ for other patterns.
- Joint distributions of patterns.

References

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Thanks for listening!

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