

Scaling limits of permutation classes with a finite specification: a dichotomy

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Permutation Patterns 2019

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Joint work with Frédérique Bassino, Mathilde Bouvel,
Valentin Féray, Lucas Gerin and Mickaël Maazoun

Main issue of this talk

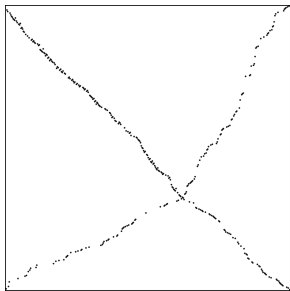
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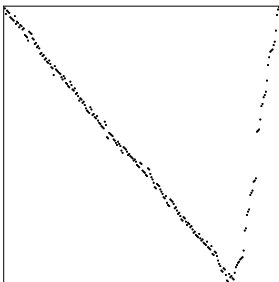


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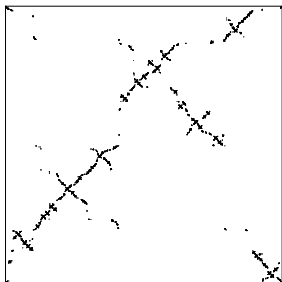
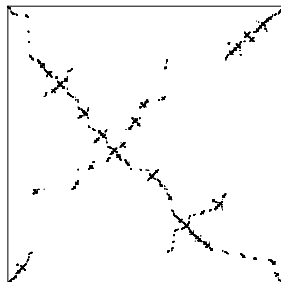


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Large permutations of $\text{Av}(2413, 3142)$ (separable permutations).

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How do typical large permutations in $\mathcal{C}$ look like?

σ_n a uniform random permutation in $\mathcal{C}$ of size $n \rightarrow \infty$

→ Limit shape of the diagram of σ_n ?

→ Frequency of occurrence of patterns in σ_n ?

Pattern densities

Frequency of occurrence of patterns:

$$\widetilde{\text{occ}}(\pi, \sigma) = \frac{\text{number of occurrences of } \pi \text{ in } \sigma}{\binom{n}{k}} \quad \text{for } n = |\sigma|, k = |\pi|$$

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- asymptotics of $\mathbb{E}[\widetilde{\text{occ}}(\pi, \sigma_n)]$?
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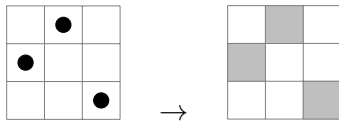
→ linked with limit shapes thanks to [permutons](#)

Permutons

A **permuton** μ is a **probability measure on $[0, 1]^2$** such that (x, y) drawn under $\mu \Rightarrow x$ (resp. y) is uniform on $[0, 1]$.

Permutation $\sigma \Rightarrow$ permuton μ_σ :

normalize the diagram and fill in uniformly cells containing dots



- Permuton **approximate** permutation diagrams
- A sequence (σ_n) converges to a permuton μ when $\mu_{\sigma_n} \xrightarrow{n \rightarrow \infty} \mu$ for the weak convergence of measures.
- σ_n random permutation $\Rightarrow \mu_{\sigma_n}$ **random permuton**

Patterns in permutons

- σ a permutation

$$\widetilde{\text{occ}}(\pi, \sigma) = \frac{\# \text{ occurrences of } \pi \text{ in } \sigma}{\binom{n}{k}} = \mathbb{P}(\text{pat}_{\mathbf{I}}(\sigma) = \pi)$$

with $\mathbf{I}$ a uniform random subset of $[n]$ with k elements

- μ a permuton

$\widetilde{\text{occ}}(\pi, \mu)$ = probability that k points drawn from μ
are isomorphic to the diagram of π

Random permutons convergence

Theorem:[Bassino-Bouvel-Gerin-Féray-Maazoun-Pierrot '17]:

(σ_n) random permutations of size n . The following are **equivalent**:

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Then $(\Lambda_\pi)_\pi \stackrel{d}{=} (\widetilde{\text{occ}}(\pi, \mu))_\pi$ and $\forall \pi \in \mathfrak{S}, \Delta_\pi = \mathbb{E}[\widetilde{\text{occ}}(\pi, \mu)]$

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Find the **permuton limit** of (σ_n) uniform random permutations of size $n \rightarrow \infty$ in a **permutation class**.

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Key tool:

substitution decomposition of permutations.

Permutations $\Leftrightarrow$ trees

Theorem[Albert Atkinson 05]:

Every permutation $\sigma \neq 1$ can be uniquely decomposed as either:

- $\alpha[\pi^{(1)}, \dots, \pi^{(d)}]$ where α is simple of size $d \geq 4$
- $12[\pi^{(1)}, \pi^{(2)}]$ where $\pi^{(1)}$ is 12-indecomposable
- $21[\pi^{(1)}, \pi^{(2)}]$ where $\pi^{(1)}$ is 21-indecomposable

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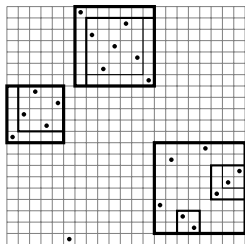
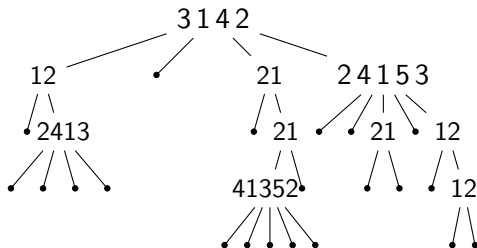
Bijection: permutation $\sigma \leftrightarrow$ standard tree T_σ :

$$\sigma = \alpha[\pi^{(1)}, \dots, \pi^{(d)}] \Leftrightarrow T_\sigma = \begin{array}{c} \alpha \\ \swarrow \quad \searrow \\ T_1 \quad T_2 \quad \cdots \quad T_d \end{array} \quad \text{where } T_i = T_{\pi^{(i)}}$$

Standard tree: rooted planar tree s.t.

- internal nodes are labeled by a simple permutation
- a node labeled by α has degree $|\alpha|$
- the left-child of a node labeled by 12 cannot be labeled by 12 (idem for 21).

Example



Example: $\sigma = 10\ 12\ 14\ 11\ 13\ 1\ 21\ 19\ 16\ 18\ 20\ 17\ 15\ 4\ 8\ 3\ 2\ 9\ 5\ 6\ 7$
 $= 3142\ [13524, 1, 7524631, 37218456]$
 $= 3142\ [12[1, 2413], 1, 21[1, 524631], 24153[1, 1, 21, 1, 123]]$
 $= \dots$

Closed classes $\rightarrow$ Specification

If $\mathcal{C}$ is a substitution-closed class (containing 12, 21)

$$\mathcal{C} = \{1\} \uplus 12[\mathcal{C}^+, \mathcal{C}] \uplus 21[\mathcal{C}^-, \mathcal{C}] \uplus \biguplus_{\pi \in \mathcal{S}_{\mathcal{C}}} \pi[\mathcal{C} \dots \mathcal{C}]$$

Where $\mathcal{S}_{\mathcal{C}} = \{\pi \in \mathcal{C} \mid \pi \text{ simple } \neq 12, 21\}$

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$\hookrightarrow$ **Combinatorial Specification**

(Recursive description using combinatorial constructors)

Tree-specification

Theorem[Bassino-Bouvel-Pierrot-Pivoteau-Rossin '17]:

Any class $\mathcal{C}$ with finitely many simple permutations admits a combinatorial specification of the form

$$\mathcal{C}_i = \varepsilon_i\{\bullet\} \uplus \biguplus_{\alpha \in \mathcal{S}_{\mathcal{C}_i} \cup \{12, 21\}} \biguplus_{(k_1, \dots, k_{|\alpha|}) \in K_{\alpha}^i} \alpha[\mathcal{C}_{k_1}, \dots, \mathcal{C}_{k_{|\alpha|}}] \quad (0 \leq i \leq d)$$

where $\varepsilon_i \in \{0, 1\}$ and $\mathcal{C} = \mathcal{C}_0 \supset \mathcal{C}_1, \dots, \mathcal{C}_d$.

The specification can be computed **algorithmically**.

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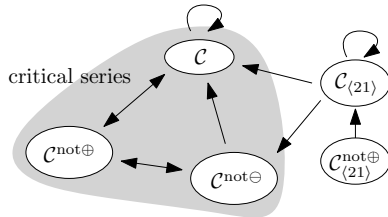
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Example: $Av(132)$

$$\left\{ \begin{array}{lcl} \mathcal{C} & = \{\bullet\} & \uplus 12[\mathcal{C}^{\text{not}\oplus}, \mathcal{C}_{\langle 21 \rangle}] \uplus 21[\mathcal{C}^{\text{not}\ominus}, \mathcal{C}] \\ \mathcal{C}^{\text{not}\oplus} & = \{\bullet\} & \uplus 21[\mathcal{C}^{\text{not}\ominus}, \mathcal{C}] \\ \mathcal{C}^{\text{not}\ominus} & = \{\bullet\} & \uplus 12[\mathcal{C}^{\text{not}\oplus}, \mathcal{C}_{\langle 21 \rangle}] \\ \mathcal{C}_{\langle 21 \rangle} & = \{\bullet\} & \uplus 12[\mathcal{C}_{\langle 21 \rangle}^{\text{not}\oplus}, \mathcal{C}_{\langle 21 \rangle}] \\ \mathcal{C}_{\langle 21 \rangle}^{\text{not}\oplus} & = \{\bullet\} & \end{array} \right.$$

Associated dependency graph:



In grey: families with **maximal growth rate** (called **critical families**)

Main Theorem

Theorem[Bassino-Bouvel-Gerin-Féray-Maazoun-Pierrot '19]:

$\mathcal{C}$ class of permutations with an **analytic tree-specification** s.t. the **dependency graph restricted to critical families is strongly connected** (plus some weak aperiodicity assumption).

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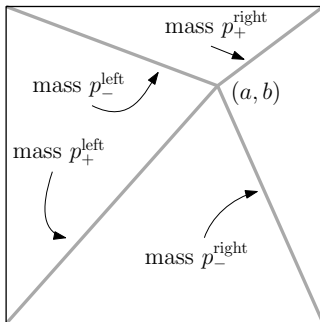
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If the specification contains **no products of critical families** (**essentially linear case**), then $\sigma_n \xrightarrow{n \rightarrow \infty}$ an **X-permutor** with computable parameters.

If the specification contains **a product of critical families** (**essentially branching case**), then $\sigma_n \xrightarrow{n \rightarrow \infty}$ a **Brownian separable permutor** with computable parameter.

The X-permutor

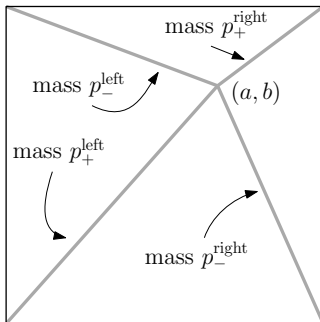
parameters $(p_+^{\text{left}}, p_+^{\text{right}}, p_-^{\text{left}}, p_-^{\text{right}}) \in [0, 1]^4$ with sum 1



$$a = p_+^{\text{left}} + p_-^{\text{left}}$$
$$b = p_+^{\text{left}} + p_-^{\text{right}}$$

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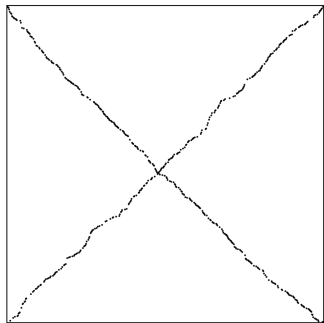
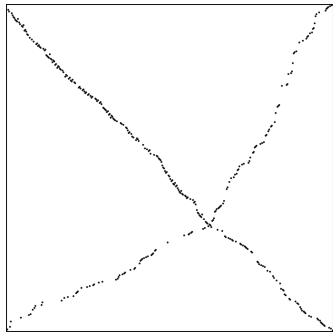
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This is a **deterministic** permutor; $\sigma_n \rightarrow \text{X-permutor} \Rightarrow$ **concentration** phenomenon (independent random permutations are closed to each other)

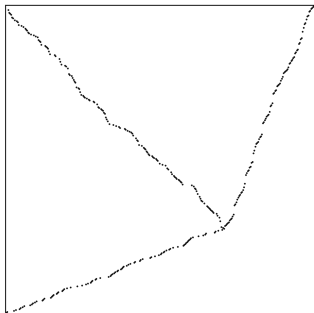
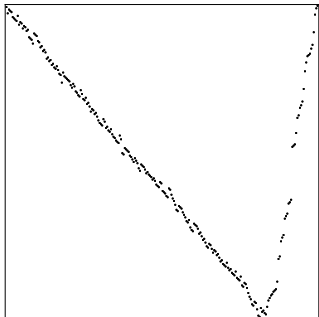
Essentially linear case: Examples



Large permutation of:

- left: $Av(2413, 3142, 2143, 3412) \rightarrow \text{parameter } (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
- right: $Av(2413, 3142, 2143, 34512) \rightarrow \text{parameter } \approx (0.2003, 0.2003, 0.4313, 0.1681)$

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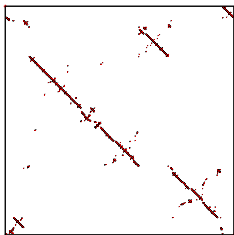
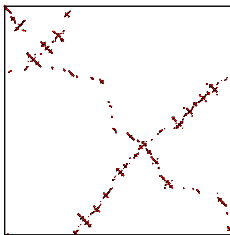
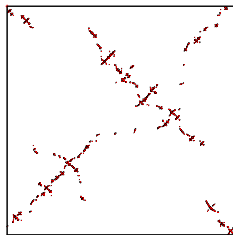


Large permutation of:

- left: $Av(231, 2143) \rightarrow \text{parameter } (a, b, c, 0)$
- right: $Av(2413, 1243, 2341, 41352, 531642) \rightarrow \text{parameter } (0, 0, 1 - p, p) \text{ with } p \approx 0.81863$

The Brownian separable permuton μ^p

parameter $p \in [0, 1]$



Simulations of μ^p for $p = 0.2$, $p = 0.45$ and $p = 0.5$
 p higher $\Leftrightarrow$ more drift towards the main diagonal

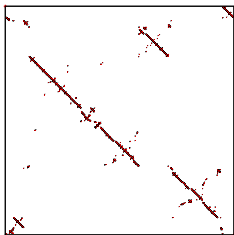
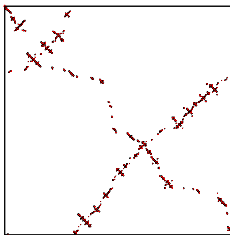
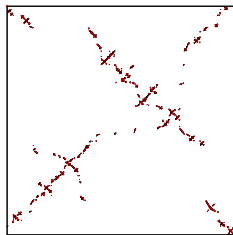
μ^p characterized by

$$\text{perm}_k(\mu^p) \stackrel{d}{=} \text{perm}(\mathbf{b}_k), \text{ for every } k \geq 1$$

where $\mathbf{b}_k$ uniform random binary tree with k leaves, with i.i.d. signs of bias p (probability of having a $\oplus$ on internal nodes).

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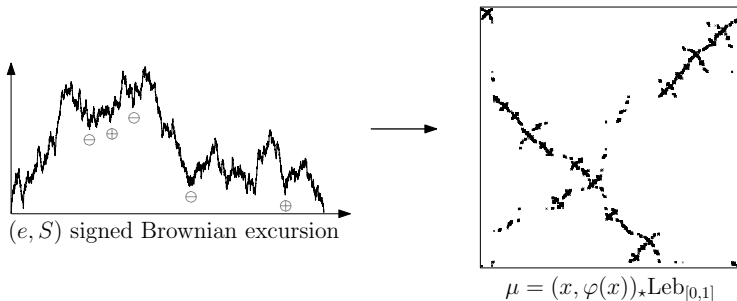
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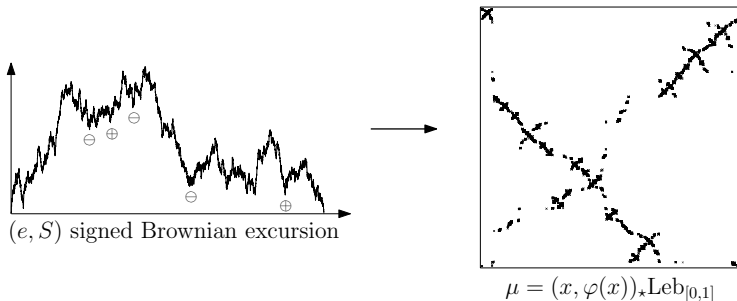
$\mu^{(p)}$ is a random permuton. No concentration phenomenon here.

Construction of the Brownian separable permuton μ^p



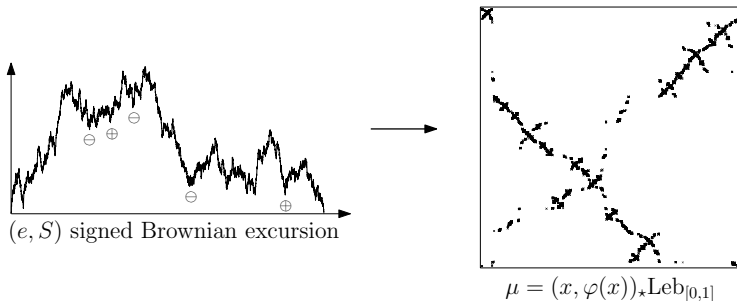
e is a Brownian excursion and $S : \text{LocalMin}(e) \rightarrow \{\oplus, \ominus\}$ an independent assignment of signs to local minima of e (the probability to get a $\oplus$ is p).

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 (x, y) is an inversion for $\varphi \Leftrightarrow$ the sign of $\min_{[x,y]} e$ is $\ominus$.

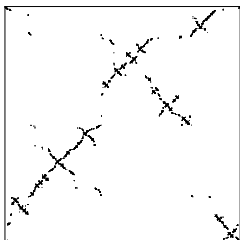
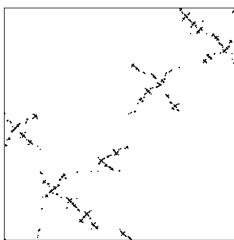
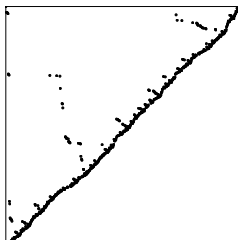
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The Brownian separable permuton is the "graph of the function φ ".
 (i.e., the pushforward of the Lebesgue measure on $[0, 1]$ by the
 function $x \rightarrow (x, \varphi(x))$).

Essentially branching case: Examples



Large permutation of:

- left: $Av(2413, 3142)$ (separable permutations): $p = 0.5$
- center: $Av(2413, 31452, 41253, 41352, 531642)$: $p \approx 0.47$
- right: $Av(231)$: $p = 1$

Sketch of proof

σ_n = uniform random permutation of size n in $\mathcal{C}$

- Reminder: enough to prove that, for any π ,
 $\mathbb{E}[\widetilde{\text{occ}}(\pi, \sigma_n)] \rightarrow \mathbb{E}[\widetilde{\text{occ}}(\pi, \mu)]$
where μ is the targeted limit random permutation.
- RHS evaluated easily
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- Numerator and denominator are expressed as **coefficients of generating series** of trees.
- The combinatorial specification for $\mathcal{C}$ yields equations for these tree series.
- Estimate their coefficients with **analytic combinatorics**: singularity analysis and transfer theorem.

Sketch of proof: analytic combinatorics

The strongly connectedness hypothesis ensures that in the **essentially linear case**,

$$C(z) \sim a \frac{1}{1 - \frac{z}{\rho}}, \text{ implying } |C_n| \sim a \rho^n$$

in the **branching case**,

$$C(z) \sim a - b \sqrt{1 - \frac{z}{\rho}}, \text{ implying } |C_n| \sim \frac{b}{2\sqrt{\pi}} n^{3/2} \rho^n$$

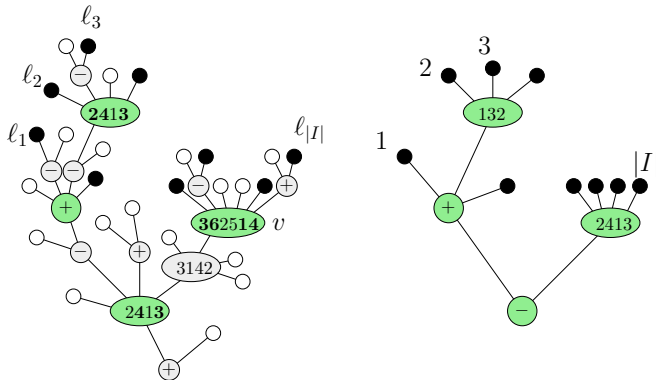
The difficulty is to estimate $|\{\sigma \in \mathcal{C}, I \subset [n] : \text{pat}_I(\sigma) = \pi\}|$

express it as coefficients of generating series of trees with **marked leaves**

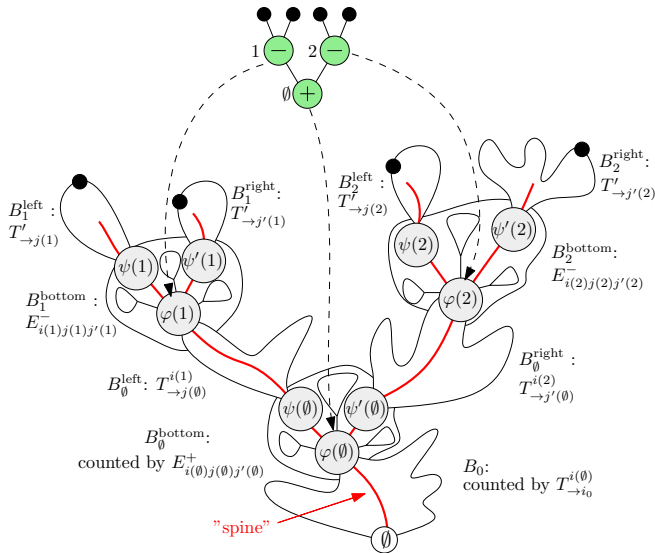
write equations for the corresponding generating function

find the behavior at the singularity

Sketch of proof: patterns in trees



Sketch of proof: combinatorial decomposition



Beyond the strongly connected case

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Case-by-case study (some lemmas but no general theorem).

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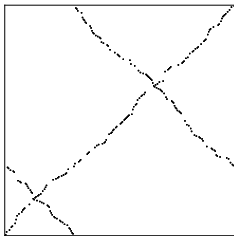
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Sometimes, the limit shape of $\mathcal{C}$ is a **combination** of those in each component.

Example: $Av(214365, 3412, 52143, 32541)$:



Summary of the described limit shapes

(Biased) Brownian separable permuton:

- Classes with a specification that is essentially strongly connected and essentially branching (includes substitution-closed classes).

(Parametrized) X-permuton:

- Classes with a specification that is essentially strongly connected and essentially linear.

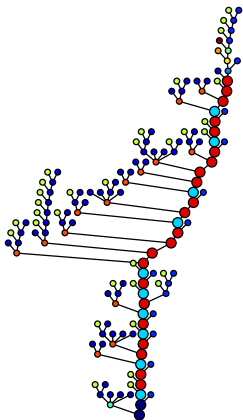
A mix of the above:

- Classes with a specification that is not essentially strongly connected.

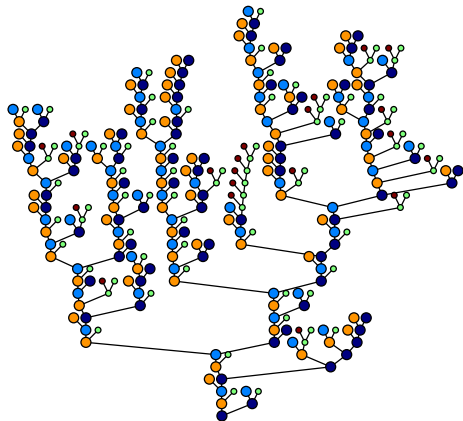
All these results need analyticity conditions (trivially satisfied when the number of simples in the class is finite).

For substitution-closed classes, we can go a bit further.

Substitution trees

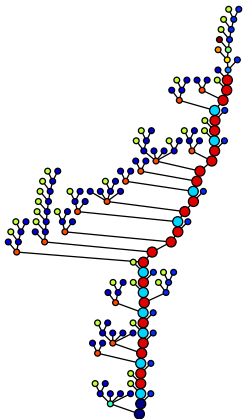


Essentially linear case

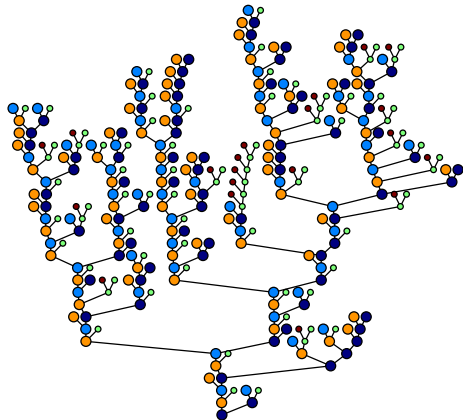


Essentially branching case

Substitution trees



Essentially linear case



Essentially branching case

Thank you!