

From presheaves to Hopf algebras

Permutation Patterns 2019, Zurich

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18th June, 2019

Slides can be found at

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Counting occurrences of a pattern

Permutation π on a set S (a pair of orders on S).

I subset of S - set of columns of the square configuration of π .

The **restriction to I** can be defined $\pi|_I$ and is a permutation in I .

We can count occurrences!

We write

$$\mathbf{p}_{12}(132) = 2, \mathbf{p}_{123}(123456) = 20, \mathbf{p}_{2413}(762341895) = 0.$$

Permutation pattern algebra

Set of permutations in $[n]$ for $n \geq 0$ - $\mathcal{G}(\text{Per})$

Pattern function p_π are in the space of functions $\mathcal{F}(\mathcal{G}(\text{Per}), \mathbb{R})$

The linear span of all pattern functions - $\mathcal{A}(\text{Per})$ - is an algebra (closed for multiplication).

Adding another ingredient

$$\pi \oplus \tau = \begin{array}{|c|c|} \hline & \tau \\ \hline \pi & \\ \hline \end{array} \quad \pi \ominus \tau = \begin{array}{|c|c|} \hline \pi & \\ \hline & \tau \\ \hline \end{array}$$

By *magic properties* of dualization, these give coproducts on $\mathcal{A}(\text{Per})$, for instance:

$$\Delta \mathbf{p}_\pi = \sum_{\pi = \tau_1 \oplus \tau_2} \mathbf{p}_{\tau_1} \otimes \mathbf{p}_{\tau_2},$$

so that we have a Hopf algebra

$$\mathbf{p}_\pi(\sigma_1 \oplus \sigma_2) = \Delta \mathbf{p}_\pi(\sigma_1 \otimes \sigma_2).$$

Permutation pattern algebra

Proposition (Linear independence)

The set $\{\mathbf{p}_\pi \mid \pi \in \uplus_{n \geq 0} S_n\}$ is linearly independent - Triangularity argument

Proposition (Product formula)

Let $\binom{\sigma}{\pi, \tau}$ count the number of covers of σ with permutations π, τ .

$$\mathbf{p}_\pi \cdot \mathbf{p}_\tau = \sum_{\sigma} \binom{\sigma}{\pi, \tau} \mathbf{p}_\sigma,$$

where σ runs over equivalence classes of pairs of orders.

Theorem (Vargas, 2014)

*The Hopf algebra $\mathcal{A}(\text{Per})$ is free comutative. **what is free?***

Outline of the talk

- 1 Introduction
 - Permutations
 - Combinatorial presheaves
- 2 Free pattern Hopf algebras
 - Cocommutative pattern Hopf algebras
- 3 Non-cocommutative examples
 - Permutations
 - Marked permutations
- 4 Conclusion

Pattern algebra

What do we need to have a pattern Hopf algebra?

- 1 Assignment $S \mapsto h[S] = \{\text{structures over } S\} + \text{notion of relabelling}.$
- 2 For any inclusion $V \hookrightarrow W$, a restriction map $h[W] \rightarrow h[V].$
- 3 An associative *monoid* operation $*$ with unit, in $\mathcal{G}(\mathbf{h})$ that is compatible with restrictions.
- 4 A unique element of size zero.

$1 + 2 = \text{combinatorial presheaf} \rightarrow \text{Algebra}.$

$1 + 2 + 3 = \text{monoid in combinatorial presheaves}.$

$1 + 2 + 4 = \text{connected presheaf}.$

$1 + 2 + 3 + 4 \rightarrow \text{Hopf algebra}.$

Category theory formulation

Observation: The product structure on $\mathcal{A}(\mathbf{h})$ depends only on the **combinatorial presheaf structure**, and **not** on the **monoid structure**.

The same product structure may be compatible with several coproducts.

Examples: the presheaves of **marked graphs** or **permutations**.

We have a functor $\mathcal{A}$ that sends

$$\mathcal{A} : \mathbf{CPSh} \rightarrow \mathbf{Alg}_{\mathbb{R}},$$

and restricts $\mathcal{A} : \mathbf{Mon}(\mathbf{CPSh}) \rightarrow \mathbf{Hopf}_{\mathbb{R}}$.

A presheaf on graphs

- For each set V we are given the set $\mathcal{G}[V]$ of graphs with vertex set V , and for any bijection $\phi : V \rightarrow W$ we are given a relabelling of graphs $\mathcal{G}[W] \rightarrow \mathcal{G}[V]$.
- Induced subgraphs $\rightarrow$ restrictions.
- The disjoint union of graphs is an associative monoid structure. It is also **commutative**.
- The empty graph fortunately exists!

Theorem (P - 2019+)

If $\mathfrak{h}$ is a connected commutative presheaf, then $\mathcal{A}(\mathfrak{h})$ is free. The free generators are the indecomposable objects with respect to the commutative product.

Simple example

The presheaf of sets.

For each $n \geq 0$, we have a unique element $*_n$ of size n up to isomorphism.

$$\mathbf{p}_{*_n}(*_m) = \binom{m}{n} \quad \binom{*_d}{*_a, *_b} = \binom{d}{a} \binom{a}{a+b-d}.$$

So

$$\mathbf{p}_{*_a} \mathbf{p}_{*_b}(*_c) = \sum_{d \geq 0} \binom{d}{a} \binom{a}{a+b-d} \mathbf{p}_{*_d}(*_c)$$

Product rule - Disjoint union: $*_n *_m = *_{n+m}$.

$$\Delta \mathbf{p}_{*_a} = \sum_{k=0}^a \mathbf{p}_{*_k} \otimes \mathbf{p}_{*_{a-k}}, \quad \mathcal{A}(\mathbf{Set}) = \mathbb{R}\langle \mathbf{p}_{*_1} \rangle$$

Connected commutative combinatorial presheafs

Proof (by example):

Graphs, with a disjoint union, form a commutative presheaf. Every graph has a unique factorisation into **indecomposables** $\mathcal{I}$.

$$\mathcal{A}(G) \text{ is free commutative} \Leftrightarrow \left\{ \prod_{l \in L} \mathbf{p}_l \mid L \subseteq \mathcal{I} \text{ multiset} \right\} \text{ is lin. ind.}$$

$$\Leftrightarrow_{\substack{\text{triangularity} \\ \text{argument}}} \prod_{l \in L} \mathbf{p}_l = \mathbf{p}_\alpha + \sum_{\beta \leq \alpha} c_\beta \mathbf{p}_\beta \text{ for some order } \leq .$$

where $\alpha = \bigsqcup_{l \in L} l$.

Highly important: **We have a unique factorisation theorem on graphs.**

To remember: A unique factorisation theorem up to commutativity of factors $\Rightarrow$ an order to use the triangularity argument.

Unique factorisation theorem on permutations

Vargas used the $\oplus$ product on permutations to obtain a unique factorisation theorem on permutations.

$$\pi = \tau_1 \oplus \cdots \oplus \tau_k =$$

		τ_k
	$\ddots$	
τ_1		

- The factorisation is **not** unique up to order of factors.

Enlarge the set $\mathcal{I}$ to $\mathcal{L}$ with **Lyndon permutations**, by adding some decomposable elements. Choose between $\pi_1 \oplus \pi_2$ and $\pi_2 \oplus \pi_1$, and between more factors.

Lyndon words - used for the freeness of the shuffle algebra on $\mathbb{K}\langle A \rangle$.

The inflation product - Marked permutations

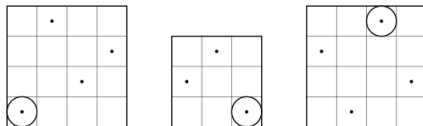
Presheaf of marked permutations - use the inflation product.

$$\pi = \begin{array}{|c|c|c|} \hline & \odot & \\ \hline & & \cdot \\ \hline \cdot & & \\ \hline \end{array}, \quad \sigma_1 = \begin{array}{|c|c|} \hline \cdot & \\ \hline & \odot \\ \hline \end{array}$$

Inflation of $\pi * \sigma$ is

		•	
		⊙	
			•
•			

Examples of indecomposable marked permutations:



Unique factorisation theorem on marked permutations

- The factorisation is **not** unique up to order of factors.
- The order of the factors does matter **only to some extent**.

The inflation map is a morphism of monoids $*$: $\mathcal{W}(\mathcal{I}) \rightarrow \mathcal{A}(\text{MPer})$.
 If τ_1, τ_2 $\oplus$ -indecomposable.

$$(\bar{1} \oplus \tau_1) * (\tau_2 \oplus \bar{1}) = (\tau_2 \oplus \bar{1}) * (\bar{1} \oplus \tau_1) = \tau_2 \oplus \bar{1} \oplus \tau_1.$$

For $\tau_1 = 2413$ and $\tau_2 = 21$ we have

$$(\bar{1} \oplus \tau_1) * (\tau_2 \oplus \bar{1}) = 21 \oplus \bar{1} \oplus 2413 =$$

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		⊙				
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Unique factorisation theorem on marked permutations

Monoid morphism $*$: $\mathcal{W}(\mathcal{I}) \rightarrow \mathcal{A}(\text{MPer})$

$$\oplus\text{-relations} : (\bar{1} \oplus \tau_1) * (\tau_2 \oplus \bar{1}) = (\tau_2 \oplus \bar{1}) * (\bar{1} \oplus \tau_1) = \tau_2 \oplus \bar{1} \oplus \tau_1 .$$

Theorem (P - 2019+)

*The equivalence relation $\ker *$ is spanned by relations as the one above and their $\ominus$ equivalent.*

Further questions - *Permutons*

Permutons - Notion of patterns of a permutation π can be extended to a permuton P : $\mathbf{p}_\pi(P) = \mathbb{P}[n \text{ i.i.d. points with law } P \text{ form pattern } \pi]$.

Conjecture

Let $\mathcal{L}_q = \{\mathbf{p}_l \mid l \text{ is a Lyndon permutation with size } \leq q\}$ be the set of free generators of $\mathcal{A}(\text{Per})$. The image of the map

$$\prod_{l \in \mathcal{L}_q} \mathbf{p}_l : \{\text{Permutons}\} \rightarrow \mathbb{R}^{\#\mathcal{L}_q},$$

is full dimensional.

Partial results for the map $\prod_{l \in \mathcal{I}} \mathbf{p}_l$ by Kenyon, Krall et al.

Further questions - *Algebra*

- *Character Theory*: simple characters can be constructed. All these have "compact support":

$$\zeta_a(\mathbf{p}_b) = \mathbf{p}_b(a),$$

and all its convolutions. Can we describe all characters? Are these all "compactly supported characters" of a free pattern algebra?

- *Freeness*: Are pattern algebras free in general? Other examples include set compositions, polyominoes, etc.
- *Other monoidal products*: The "marked" operations come from the *operad* monoidal structure on permutations. *Hadamard product*? *Heisenberg product*?

Biblio

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Thank you

