



Consecutive permutation patterns in trees and mappings

Alois Panholzer

Institut für Diskrete Mathematik und Geometrie

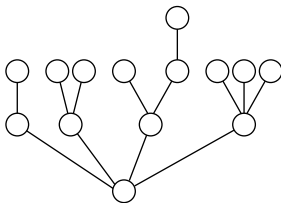
TU Wien, Austria

Alois.Panholzer@tuwien.ac.at

Permutation Patterns 2019, Universität Zürich, 20.06.2019

Label quantities in trees

Rooted tree: problems where labels of nodes play important rôle

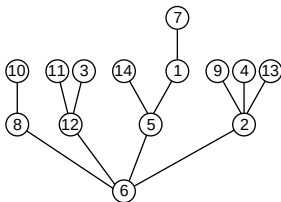


Examples of label quantities in trees:

- **Leader in trees:** node smallest label amongst labels in subtree
Gessel & Seo 2006; Seo 2007; Hou 2016
→ combinatorial proof of hook-length formulas for trees
- **Record in trees:** node smallest label amongst path to root
P. 2006; Janson 2006; Goldschmidt & Martin 2005; ...
→ “cutting-down trees”, relations to coalescence models

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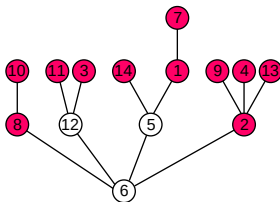


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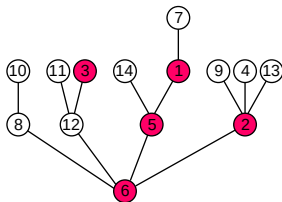


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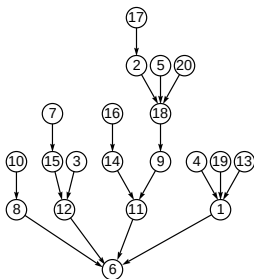


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Classical permutation patterns in trees

Consider edges of tree oriented towards root node

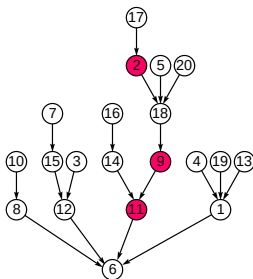
 x_1 $\downarrow$ x_2 $\downarrow$ $\vdots$ $\downarrow$ x_k

Occurrence of classical permutation pattern $s = s_1, \dots, s_k \in S_k$:

sequence $x_1, \dots, x_k$ of labels on path to root node that are order-isomorphic to s

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Known results for classical permutation patterns in trees

Occurrence of classical permutation patterns:

- **pattern 21**: “inversions in trees”
Mallows & Riordan 1968; P. & Seitz 2012; Cai, Holmgren, Janson, Johansson & Skerman 2019
→ studied for various tree families
- **general patterns**
Albert, Holmgren, Johansson & Skerman 2018
→ limiting distribution results: full binary trees and split trees

Avoidance of classical permutation patterns:

- **pattern 21**: “increasing trees”
large number of works in combinatorics, probability, etc.
- **sets of patterns of length 3**
Anders & Archer 2019 → explicit enumeration formulæ for unordered trees

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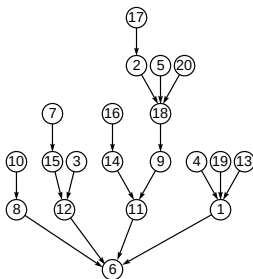
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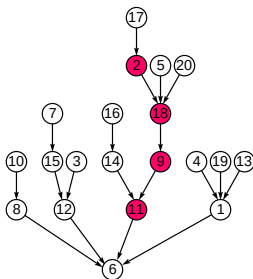
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Occurrence of consecutive permutation pattern $s = s_1, \dots, s_k \in S_k$:

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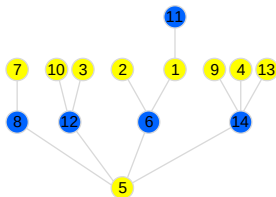
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Known results for c-permutation patterns in trees

- Occurrence of 21: “descents in trees”
Eğecioğlu & Remmel 1986; Clark 2014
→ exact formulæ for p.g.f. and limiting distribution results for unordered trees
- Avoidance of set $\{123, 321\}$: “alternating trees”

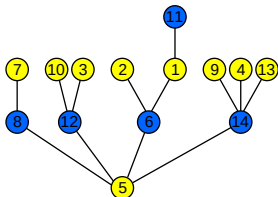


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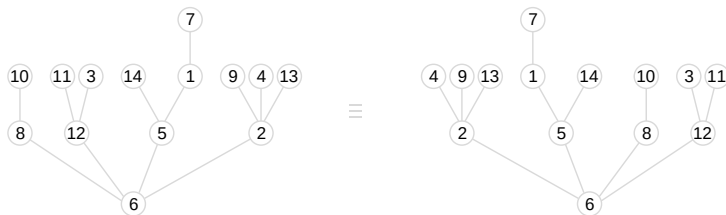
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Consecutive patterns of length 3 in trees

Present work: Study of **single consecutive permutation pattern s** of **length 3** in trees

tree model: rooted labelled unordered trees
 → no left-to-right ordering on children

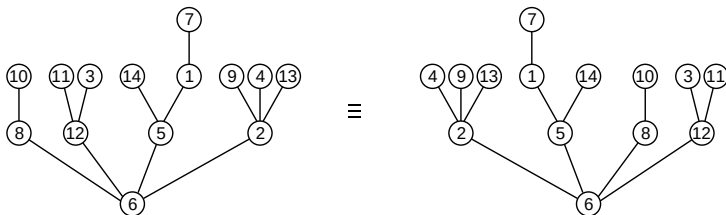


unrestricted model: $T_n = n^{n-1}$ size- n trees

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Results for consecutive permutation patterns

Avoiding a single pattern of length 3:

Theorem (P. 2019)

$T_n^{[s]}$: # size- n trees avoiding pattern s

Generating functions $T = T^{[s]}(z) := \sum_{n \geq 1} T_n^{[s]} \frac{z^n}{n!}$

Characterized via *functional equations*:

Pattern class $Av(s)$	$T := T^{[s]}(z)$
$Av(123)$	$z = e^{-T} \int_0^T \frac{e^t}{1+t} dt$
$Av(321)$	
$Av(132)$	$z = \int_0^T e^{-t-(T-t)e^{-t}} dt$
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Asymptotic behaviour

Theorem

$$T_n^{[s]} \sim c_T \cdot \gamma^n \cdot n^{n-1}$$

$\gamma = \frac{1}{e\rho}$, ρ : radius of convergence of $T^{[s]}(z)$
 c_T : computable constants

Pattern s	ρ	γ	c_T
123	0.42718536...	0.86117050...	1.53000135...
132	0.44084481...	0.83448739...	1.74299311...
231	0.44922576...	0.81891883...	2.23735314...

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Remarks

- Numbers $T_n^{[s]}$ for small n :

Pattern s	$T_n^{[s]}, n \in [8]$							
	1	2	3	4	5	6	7	8
123	1	2	8	50	426	4606	60418	932282
132	1	2	8	49	407	4280	54537	816905
231	1	2	8	49	406	4248	53740	797786

- Only $T_n^{[123]}$ occurs in OEIS: sequence A225052, without providing combinatorial interpretation.
- Compare exponential growth rates γ :

21	231	132	123	unrestricted
0.3678...	0.8189...	0.8344...	0.8611...	1

- Asymptotic relation for the enumeration sequences $T_n^{[s]}$:

$$T_n^{[231]} \ll T_n^{[132]} \ll T_n^{[123]}$$

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Occurrences of a single pattern of length 3:

Theorem (P. 2019)

$F_{n,m}^{[s]}$: # size- n trees with m occurrences of pattern s

Generating functions $F = F^{[s]}(z, v) := \sum_{n \geq 1} \sum_{m \geq 0} F_{n,m}^{[s]} \frac{z^n v^m}{n!}$

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Theorem

$X_n^{[s]}$: r.v. counting occurrences of pattern s in random size- n tree

Mean and variance:

	$\mathbb{E}(X_n^{[s]})$	$\mathbb{V}(X_n^{[s]})$
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Gaussian limiting distribution result:

$$\frac{X_n^{[s]} - \mathbb{E}(X_n^{[s]})}{\sqrt{\mathbb{V}(X_n^{[s]})}} \xrightarrow{(d)} \mathcal{N}(0, 1)$$

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Method of proof

Outline:

- Combinatorial decomposition w.r.t. vertex 1
- Requires taking into account additional quantities depending on considered pattern
- Generating functions approach for trivariate g.f.
 - 123, 132: quasi-linear first-order PDE
 - 231: ODE, but function also evaluated at “diagonal”
- Solving via method of characteristics
- Eliminating auxiliary quantity
 - $F(z, v)$ & $T(z) = F(z, 0)$
- Asymptotic/Limiting distribution results via analytic combinatorics

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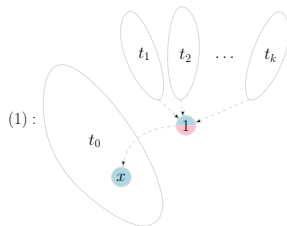
Proof for pattern 123

- **Decomposition w.r.t. vertex 1:**

y coloured blue: starting node of pattern 12

y coloured red: starting node of pattern 123

Three cases:



Node 1 attached to blue node

$\Rightarrow$ new pattern 123 and 12 generated

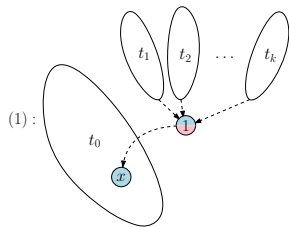
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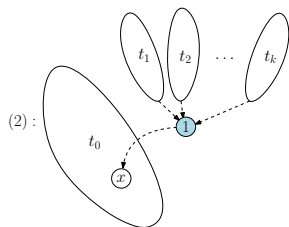
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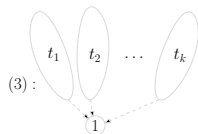
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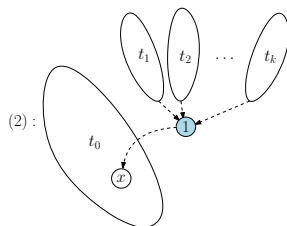


Node 1 attached to non-blue node
 $\Rightarrow$ new pattern 12 generated

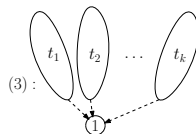


Node 1 root node $\Rightarrow$ no new patterns

Proof for pattern 123



Node 1 attached to non-blue node
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Proof for pattern 123

- Generating function:

$$\tilde{F}(z, w, v) := \sum_{t \in \mathcal{T}} \frac{z^{|t|} w^{\# \text{ blue vertices in } t} v^{\# \text{ red vertices in } t}}{|t|!}$$

- First-order quasi-linear PDE:

$$(1 - wz e^{\tilde{F}}) \tilde{F}_z + (1 - v) w^2 e^{\tilde{F}} \tilde{F}_w - e^{\tilde{F}} = 0$$

- Method of characteristics:

$$z = \int_0^{\tilde{F}} e^{-s} (1 - (v-1)w(\tilde{F}-s))^{\frac{1}{v-1}} ds$$

$$\Rightarrow F(z, v) = \tilde{F}(z, 1, v), \quad T(z) = \tilde{F}(z, 1, 0)$$

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Asymptotic considerations:

- **Study of $T(z)$ via analytic combinatorics:**
 - Implicit function theorem $\Rightarrow$ dominant singularity ρ
 - Local expansion of $T(z)$ in complex neighbourhood of ρ
 - Singularity analysis
- **Study of $F(z, v)$:**
 - "Singular perturbation": study of $F(z, v)$ for v close to 1
 - Implicit function theorem
 - Quasi-power theorem $\Rightarrow$ central limit theorem

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 - Local expansion of $T(z)$ in complex neighbourhood of ρ
 - Singularity analysis
- **Study of $F(z, v)$:**
 - "Singular perturbation": study of $F(z, v)$ for v close to 1
 - Implicit function theorem
 - Quasi-power theorem $\Rightarrow$ central limit theorem

Proof for pattern 123

Asymptotic considerations:

- **Study of $T(z)$ via analytic combinatorics:**
 - Implicit function theorem $\Rightarrow$ dominant singularity ρ
 - Local expansion of $T(z)$ in complex neighbourhood of ρ
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Proof for pattern 231

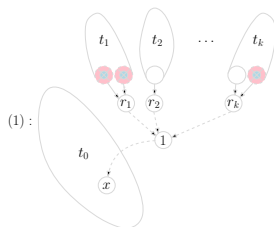
• Decomposition w.r.t. vertex 1:

y coloured blue: starting node of pattern 12 ending at root

y coloured red: starting node of pattern 231

Three cases:

In each case: every blue node in subtrees of 1 redy red



Node 1 attached to non-root node
 $\Rightarrow$ no further new patterns

Proof for pattern 231

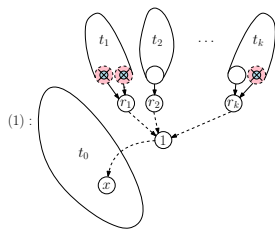
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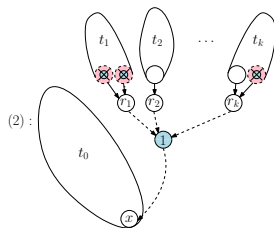
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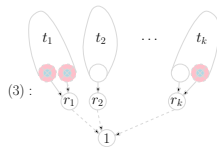
Node 1 attached to non-root node
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Proof for pattern 231



Node 1 attached to root node

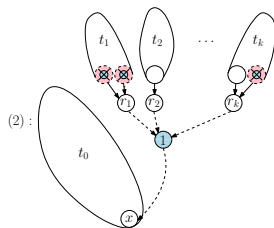
$\Rightarrow$ one further blue vertex generated



Node 1 root node

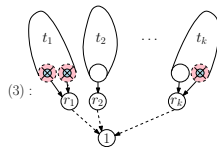
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Proof for pattern 231



Node 1 attached to root node

$\Rightarrow$ one further blue vertex generated



Node 1 root node

$\Rightarrow$ no further new patterns

Proof for pattern 231

- Generating function:

$$\tilde{F}(z, w, v) := \sum_{t \in \mathcal{T}} \frac{z^{|t|} w^{\# \text{ blue vertices in } t} v^{\# \text{ red vertices in } t}}{|t|!}$$

- ODE, function also **evaluated at** $w = v$:

$$(1 - ze^{\tilde{F}(z, v, v)}) \tilde{F}_z(z, w, v) + (1 - w)e^{\tilde{F}(z, v, v)} \tilde{F}(z, w, v) - e^{\tilde{F}(z, v, v)} = 0$$

- Solving ODE for “diagonal” first:** $A(z, v) := \tilde{F}(z, v, v)$

$$z = (1 - (1 - v)A)^{\frac{1}{1-v}} \int_0^A (1 - (1 - v)t)^{\frac{v-2}{1-v}} e^{-t} dt$$

$\Rightarrow$ Also yields solution for $F(z, v) = \tilde{F}(z, 1, v)$ and $T(z) = \tilde{F}(z, 1, 0)$

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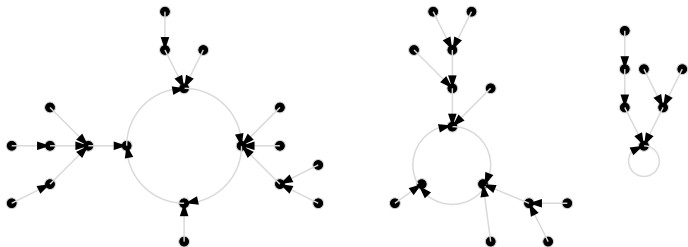
Extensions for n -mappings

n -mappings: functions f from set $[n] := \{1, 2, \dots, n\}$ into itself:

$$f : [n] \rightarrow [n]$$

Functional digraph G_f of f :

$G_f = (V_f, E_f)$, with $V_f = [n]$ and $E_f = \{(i, f(i)), i \in [n]\}$



Simple structure: connected components of G_f : cycles of trees

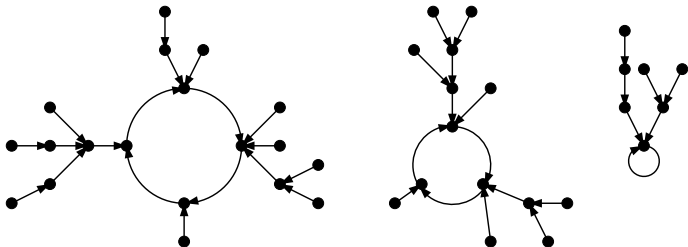
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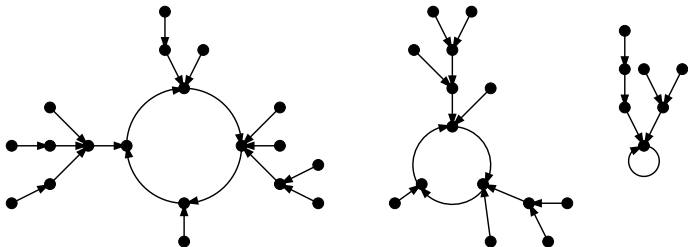
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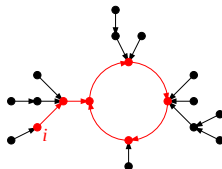
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Label patterns in mappings

Consider **iteration orbit** of elements

$\equiv$ **path** in functional digraph **starting with i** :

$$(i, f(i), f^2(i), f^3(i), \dots)$$



“Label patterns” on iteration orbit:

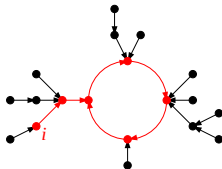
- Restrictions on the label patterns
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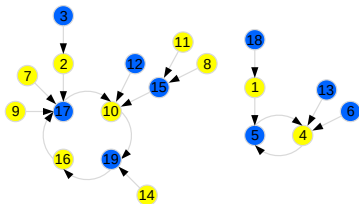
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Known results for label patterns

- **Alternating mappings** [P. 2013]:

labels on each iteration path are **up-down-alternating sequence**:



local minima alternate with **local maxima**: $\{123, 321\}$ -avoiding

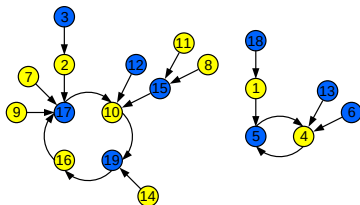
Enumeration formula: M_n : number of alternating n -mappings

$$M_n = \frac{1}{2^{n+1}} \sum_{k=0}^{n+1} \binom{n+1}{k} (k-1)^n$$

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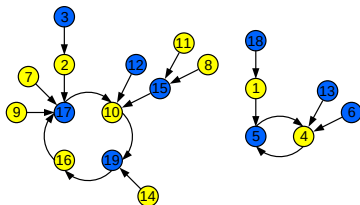
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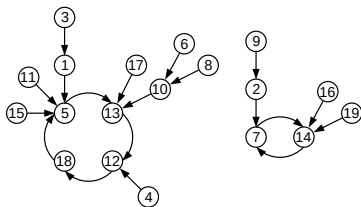
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- Ascending & descending runs in mappings:**

Ascending run: max. incr. path $i = f^0(i) < f^1(i) < f^2(i) < \dots < f^k(i)$

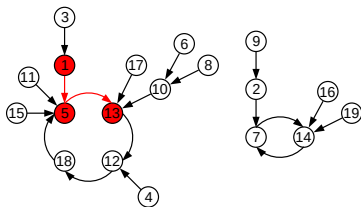


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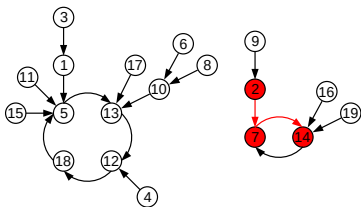


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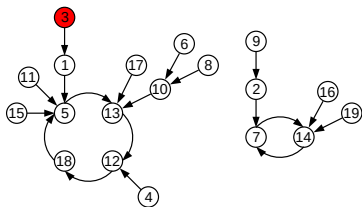


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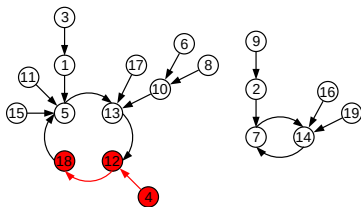


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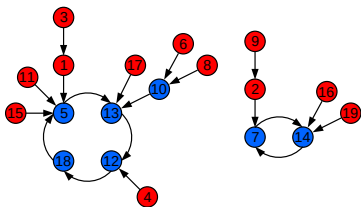


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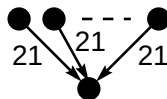
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Starting nodes of runs

$\equiv$

Local label pattern:

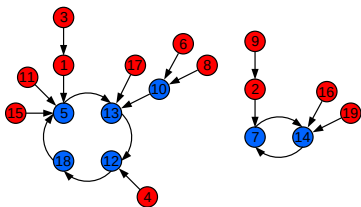


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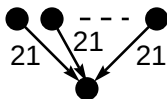
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Consecutive permutation patterns in mappings

Theorem (P. 2019)

$M_n^{[s]}$: # n -mappings *avoiding* pattern s

$G_{n,m}^{[s]}$: # n -mappings with m *occurrences* of s

$T_n^{[s]}$, $F_{n,m}^{[s]}$: corresponding *quantities for trees*

For patterns 123 and 231 it holds:

$$M_n^{[s]} = n \cdot T_n^{[s]}$$

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Pattern 132: this relation does not hold!

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Theorem

Pattern 132: *Generating functions $M(z)$, $G(z, v)$ related to corresponding g.f. for trees:*

$$M = \frac{e^{T-1+e^{-T}}}{1 - e^T \int_0^T e^{-2t-(T-t)e^{-t}} dt}$$

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Asymptotic behaviour for 132-avoiding mappings:

$$M_n \sim 1.0530 \dots \cdot n T_n$$

132-occurrences in mappings:

same limiting distribution behaviour as for trees

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Bijection proof for patterns 123 & 231

Bijection: mappings & marked labelled trees [Lackner & P. 2015+]:

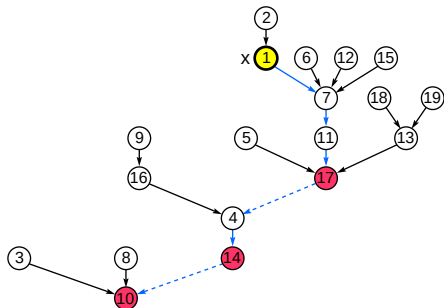
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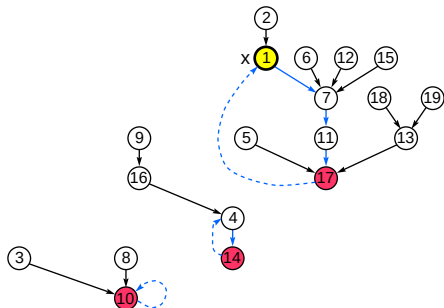


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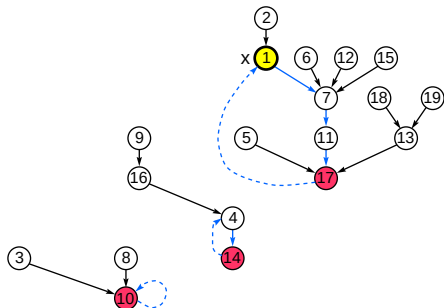
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Proof for pattern 132

Generating functions approach:

- **Decomposition** of mapping graph **w.r.t. vertex 1**
- Requires **auxiliary quantity**: occurrences of **pattern 21**
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- **Method of characteristics**
 → solution containing g.f. $\tilde{F}$ for trees

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Outlook

Possible extensions of approach:

- **Sets of patterns:**

- several interesting classes of sets of patterns of length 3
 - $Av(\{123, 321\})$: alternating trees
 - $Av(\{132, 231\})$: unimodal trees
- some (not all!) could be treated via decomposition w.r.t. largest/smallest element

- **Patterns of length 4 or higher:**

- only some patterns could be treated with this approach
- computations quickly get quite involved
- some hope to obtain results for “nice” patterns

- **Other combinatorial tree families:**

- ordered trees, binary trees: approach could be applied
- g.f. description more involved: require taking into account possible “attachment points” in decomposition

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