k-partial permutations and the center of the wreath product $\mathcal{S}_k \wr \mathcal{S}_n$ algebra

Omar Tout
Faculty of Sciences III, Lebanese University

Part 1: Definitions

k-partial permutations

If $i, k \in \mathbb{N}^*$, the k-tuple $p_k(i)$ is the following set of size k :

$$p_k(i) := \{(i-1)k+1, (i-1)k+2, \cdots, ik\}.$$

If $d = \bigsqcup_{i=1}^{r} p_k(a_i)$, where $a_i \in \mathbb{N}^*$ for any $1 \leq i \leq r$, we define $\mathcal{B}_d^k$ to be the following group of permutations:

$$\{\omega \in \mathcal{S}_d \mid \forall 1 \leq i \leq r, \exists 1 \leq j \leq r; \omega(p_k(a_i)) = p_k(a_j)\},$$

where $\mathcal{S}_d$ is the group of permutations of the set d.
A k-partial permutation of $n \in \mathbb{N}$ is a pair (d, ω) where $d \subset [kn]$ is a disjoint union of some k-tuples and $\omega \in \mathcal{B}_d^k$. $\mathcal{P}_{kn}^k$ will denote the set of all k-partial permutations of n.

Type of a k-partial permutation

A partition ρ of k is a decreasing list of positive integers $(\rho_1, \rho_2, \cdots, \rho_r)$ that sum up to k. Let $\omega \in \mathcal{B}_d^k$ be written as a product of disjoint cycles and ρ be a partition of k. Consider the collection of cycles $C_1, \ldots, C_l$ such that C_1 contains ρ_1 elements of a certain fixed k-tuple $p_k(i)$, C_2 contains ρ_2 elements $p_k(i)$, etc. Now add the part m to $\omega(\rho)$ if m is the number of k-tuples that form the cycles $C_1, \ldots, C_l$. We define type(ω) to be the family of partitions $(\omega(\rho))_{\rho \vdash k}$.

Example

Consider the 3-partial permutation (d, ω), where $d = p_3(1) \cup p_3(2) \cup p_3(4) \cup p_3(6)$ and

$$\omega = \begin{pmatrix} 1 & 2 & 3 & | & 4 & 5 & 6 & | & 10 & 11 & 12 & | & 16 & 17 & 18 \\ 12 & 10 & 11 & | & 4 & 5 & 6 & | & 16 & 18 & 17 & | & 1 & 2 & 3 \end{pmatrix}.$$

Its cycle decomposition is $(1, 12, 17, 2, 10, 16)(3, 11, 18)(4)(5)(6)$ and its type is formed by $\omega(2, 1) = (3)$ and $\omega(1^3) = (1)$.

Part 2: The algebra of k-partial permutations

The algebra of k-partial permutations

The product in $\mathcal{P}_{kn}^k$ is defined by $(d_1, \omega_1)(d_2, \omega_2) = (d_1 \cup d_2, \omega_1\omega_2)$, where the composition $\omega_1\omega_2$ is made after extending both ω_1 and ω_2 by identity to $d_1 \cup d_2$. Let $\mathcal{I}_{kn}^k$ be the algebra over $\mathbb{C}$ generated by the following formal sums indexed by families of partitions $\Lambda = (\lambda(\rho))_{\rho \vdash k}$ with $|\Lambda| := \sum_{\rho \vdash k} |\Lambda(\rho)| < n$

$$\mathbf{C}_{\Lambda;n} := \sum_{(d,\omega)} (d, \omega),$$

where the sum is taken over all $(d, \omega) \in \mathcal{P}_{kn}^k$ such that $|d| = k|\Lambda|$ and type$(\omega) = \Lambda$.

Homomorphism

There is a surjective homomorphism between the algebras $\mathcal{I}_{kn}^k$ and $Z(\mathbb{C}[\mathcal{B}_{kn}^k])$, the center of the group algebra $\mathbb{C}[\mathcal{B}_{kn}^k]$, defined by

$$\psi(\mathbf{C}_{\Lambda;n}) = \binom{n - |\Lambda| + m_1(\Lambda(1^k))}{m_1(\Lambda(1^k))} \mathbf{C}_{\underline{\Lambda}_n},$$

where $\underline{\Lambda}_n$ is the family of partitions Λ except that $\Lambda(1^k)$ is replaced by $\Lambda(1^k) \cup (1^{n-|\Lambda|})$.

A universal algebra

Let $\mathcal{P}_\infty^k$ be the group of all the k-partial permutations with a finite support and $\mathcal{I}_\infty^k$ the algebra generated by the following elements $\mathbf{C}_\Lambda = \sum_{(d,\omega)}(d, \omega)$, indexed by families of partitions, where the sum runs over all $(d, \omega) \in \mathcal{P}_\infty^k$ such that d is a union of $|\Lambda|$ k-tuples and ω has type Λ. The projection homomorphism from $\mathcal{I}_\infty^k$ to $\mathcal{I}_{kn}^k$ is defined by $\mathrm{Proj}_n(\mathbf{C}_\lambda) = 0$ if $|\Lambda| > n$ and if $|\Lambda| \leq n$, $\mathrm{Proj}_n(\mathbf{C}_\Lambda) = \mathbf{C}_{\Lambda;n}$.

Part 3: Structure coefficients of the center of $\mathcal{B}_{kn}^k$ algebra

Structure coefficients $c_{\Lambda\Delta}^\Gamma$

A family of partitions Λ is called proper if the partition $\Lambda(1^k)$ does not have any part equal to one. Let Λ and Δ be two proper families of partitions with $|\Lambda|, |\Delta| \leq n$. In the algebra $\mathcal{I}_\infty^k$, we can write

$$\mathbf{C}_\Lambda \mathbf{C}_\Delta = \sum_\Gamma c_{\Lambda\Delta}^\Gamma \mathbf{C}_\Gamma,$$

where the sum runs over the families of partitions Γ satisfying $\max(|\Lambda|, |\Delta|) \leq |\Gamma| \leq |\Lambda| + |\Delta|$ and $c_{\Lambda\Delta}^\Gamma$ are non-negative integers independent of n. If we apply $\psi \circ \mathrm{Proj}_n$ to this equality we get the following identity in the center of the group $\mathcal{B}_{kn}^k$ algebra:

$$\mathbf{C}_{\underline{\Lambda}_n} \mathbf{C}_{\underline{\Delta}_n} = \sum_\Gamma c_{\Lambda\Delta}^\Gamma \binom{n - |\Gamma| + m_1(\Gamma(1^k))}{m_1(\Gamma(1^k))} \mathbf{C}_{\underline{\Gamma}_n}.$$

The above sum can be turned into a sum over proper families of partitions.

Polynomiality Property

Theorem

Let Λ, Δ and Γ be three proper families of partitions satisfying $\max(|\Lambda|, |\Delta|) \leq |\Gamma| \leq |\Lambda| + |\Delta|$. For any integer $n \geq |\Gamma|$ we have

$$c_{\underline{\Lambda}_n \underline{\Delta}_n}^{\underline{\Gamma}_n} = \sum_{r=1}^{n-|\Gamma|} c_{\Lambda\Delta}^{\underline{\Gamma}_{|\Gamma|+r}} \binom{n - |\Gamma|}{r},$$

where $c_{\Lambda\Delta}^{\underline{\Gamma}_{|\Gamma|+r}}$ are non-negative integers independent of n.

Corollary

Let Λ, Δ and Γ be three proper families of partitions satisfying $\max(|\Lambda|, |\Delta|) \leq |\Gamma| \leq |\Lambda| + |\Delta|$. The structure coefficient $c_{\underline{\Lambda}_n \underline{\Delta}_n}^{\underline{\Gamma}_n}$ is a polynomial in n with non-negative integer coefficients and of degree at most $|\Lambda| + |\Delta| - |\Gamma|$.

Part 4: An example for $k = 2$

The elements generating $\mathcal{I}_\infty^2$ are indexed by families of partitions $\Lambda = (\Lambda(1^2), \Lambda(2))$. For instance, $C_{((1),(2))}$ is the set of all 2-partial permutations with type $((1), (2))$ and $(\{3, 4, 7, 8, 9, 10\}, (3, 7, 4, 8)(9)(10)) \in C_{((1),(2))}$. In $\mathcal{I}_\infty^2$:

$$\mathbf{C}_{((1),\emptyset)} \mathbf{C}_{((1),(1))} = 2\mathbf{C}_{((1),(1))} + 2\mathbf{C}_{((1^2),(1))} \text{ and } \mathbf{C}_{(\emptyset,(2))} \mathbf{C}_{(\emptyset,(2))} = 2\mathbf{C}_{((1^2),\emptyset)} + 2\mathbf{C}_{(\emptyset,(1^2))} + 2\mathbf{C}_{(\emptyset,(2^2))} + 3\mathbf{C}_{((3),\emptyset)}.$$

The above red coefficient is due to the fact that there are only two pairs $(x, y) \in C_{((1),\emptyset)} \times C_{((1),(1))}$ that satisfy $xy = (\{1, 2, 3, 4\}; (1)(2)(3, 4))$. Namely (x, y) can be

$$\Big(\big(\{1, 2\}; (1)(2)\big), \big(\{1, 2, 3, 4\}; (1)(2)(3, 4)\big)\Big) \text{ or } \Big(\big(\{3, 4\}; (3)(4)\big), \big(\{1, 2, 3, 4\}; (1)(2)(3, 4)\big)\Big).$$

Apply now $\psi \circ \mathrm{Proj}_n$ for the above two equations to get the following equations in $Z(\mathbf{C}[\mathcal{B}_{2n}^2])$:

$$\mathbf{C}_{((1^n),\emptyset)} \mathbf{C}_{((1^{n-1}),(1))} = \mathbf{C}_{((1^{n-1}),(1))} \text{ for any } n \geq 3$$

and for any $n \geq 5$

$$\mathbf{C}_{((1^{n-2}),(2))} \mathbf{C}_{((1^{n-2}),(2))} = n(n-1)\mathbf{C}_{((1^n),\emptyset)} + 2\mathbf{C}_{((1^{n-2}),(1^2))} + 2\mathbf{C}_{((1^{n-4}),(2^2))} + 3\mathbf{C}_{((1^{n-3},3),\emptyset)}.$$

The first equation comes with no surprise since $C_{((1^n),\emptyset)}$ is the identity class.

References

1- O. Tout, *The center of the wreath product of symmetric groups algebra*, To appear in Algebra and Discrete Mathematics (2019).
2- O. Tout, *k-partial permutations and the center of the wreath product $\mathcal{S}_k \wr \mathcal{S}_n$ algebra*, arXiv e-prints (2019).
3- V. Ivanov, S. Kerov, *The algebra of conjugacy classes in symmetric groups, and partial permutations*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 256(3)(1999) 95–120.