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Applications of a q -analog for Riordan arrays to various combinatorial objects

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Outline

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 - q -analog in combinatorics
 - q -analog in orthogonal polynomials
- 2 q -analog of Riordan arrays
 - A brief introduction to Riordan arrays
 - q -Riordan arrays
- 3 Applications
 - q -analog of Laguerre polynomials
 - Combinatorial interpretations with q -rook numbers
- 4 Concluding Remarks

I. q -analogues in several contexts

- W. A. Al-Salam, Chihara *Convolutions of orthonormal polynomials*, SIAM Journal on Mathematical Analysis 7 (1) 16-28, Theodore Seio (1976)
- A.M. Garsia, J. Remmel, *A combinatorial interpretation of q -derangement and q -Laguerre numbers*, European J. Combin. 1 (1980) 47-59.
- I.M. Gessel, *A q -analog of the exponential formula*, Discrete Math. 306 (2006), 1022-1031.
- I.M. Gessel, *Generalized rook polynomials and orthogonal polynomials*, in: D. Stanton, ed., *q -Series and Partitions* (Springer, Berlin, 1989) 154-176.

q -analog of $A \rightarrow A$ as $q \rightarrow 1$

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- q -analog of $n \in \mathbb{N}$

$$\lim_{q \rightarrow 1} \frac{q^n - 1}{q - 1} = \lim_{q \rightarrow 1} (1 + q + q^2 + \cdots + q^{n-1}) = n$$

- q -number $[n]_q = \frac{q^n - 1}{q - 1} = \sum_{k=1}^n q^{k-1}$, $[0]_q = 0$
- q -factorial $[n]_q! = \prod_{k=1}^n [k]_q$, $[0]_q! = 1$
- q -binomial coefficients $\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}$

□ q -analogs are most frequently studied in the mathematical fields of **combinatorics**, **orthogonal polynomials** and **special functions**.

- $n!$ counts the number of permutations of length n .
- $[n]_q!$ counts permutations while keeping track of the number of inversions:

$$[n]_q! = \sum_{w \in S_n} q^{\text{inv}(w)}.$$

- $\begin{bmatrix} n \\ k \end{bmatrix}_q$ counts the number of k -dimensional subspaces of the n -dimensional vector space over the q -element field.

- Explicit formula:

$$P_n(x) = \sum_{k=0}^n p_{n,k} x^k$$

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$$P_n(x) = (A_n x + B_n) P_{n-1}(x) + C_n P_{n-2}(x)$$

q -analog in orthogonal polynomials

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- Recurrence relation:

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- Generating functions:

$$P(x, t) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}, \quad \text{or} \quad P(x, t) = \sum_{n=0}^{\infty} P_n(x) t^n$$

Example: Laguerre polynomials

Laguerre polynomials $L_n^{(\alpha)}(x)$ are solutions of $xy'' + (\alpha + 1 - x)y' + ny = 0$.

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- Explicit formula

$$(1) \quad L_n^{(\alpha)}(x) = \sum_{k=0}^n (-1)^{n-k} \frac{n!}{k!} \binom{n+\alpha}{k+\alpha} x^k, \quad \alpha \in \mathbb{R}$$

$$(2) \quad L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^n \frac{(-n)_k x^k}{(\alpha+1)_k k!}, \quad (a)_k = a(a+1) \cdots (a+k-1).$$

- Recurrence relation

$$L_{n+1}^{(\alpha)}(x) = (x - (2n + \alpha + 1))L_n^{(\alpha)}(x) - n(n + \alpha)L_{n-1}^{(\alpha)}(x)$$

- Generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) \frac{t^n}{n!} = \frac{1}{(1+t)^{\alpha+1}} e^{\frac{xt}{1+t}}$$

Connection with rook polynomials

- The **rook polynomial** for a board $B \subseteq [n + \alpha] \times [n]$ where $\alpha \in \mathbb{N}_0$ and $n \in \mathbb{N}$ is defined by

$$R_B(x) = \sum_{k=0}^n (-1)^k r_k x^{n-k}$$

where r_k denotes the number of all placements of k non-attacking rooks on B .

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If B is an $(n + \alpha) \times n$ complete board, then $R_B(x) = L_n^{(\alpha)}(x)$.

q -analog of the explicit formula

- Garsia and Remmel in 1980

$$L_n^{(\alpha)}(x|q) = \sum_{k=0}^n q^{k(k+\alpha)} \frac{[n]_q!}{[k]_q!} \begin{bmatrix} n + \alpha \\ k + \alpha \end{bmatrix}_q x^k$$

$$L_n^{(\alpha)}(x|q) \rightarrow (-1)^n L_n^{(\alpha)}(-x) \text{ when } q \rightarrow 1.$$

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- Moak in 1981

$$L_n^{(\alpha)}(x; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \sum_{k=0}^n \frac{(q^{-n}; q)_k q^{\binom{k}{2}} (1-q)^k (q^{n+\alpha+1} x)^k}{(q^{\alpha+1}; q)_k (q; q)_k}$$

where $(a; q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1})$.

$$L_n^{(\alpha)}(x; q) \rightarrow L_n^{(\alpha)}(x) \text{ when } q \rightarrow 1.$$

- Kasraoui, Stanton and Zeng in 2011

$$L_n(x, y; q) = \left(\frac{\sqrt{y}}{q-1} \right)^n Q_n \left(\frac{(q-1)x + y + 1}{2\sqrt{y}}; \frac{1}{\sqrt{y}}, \sqrt{y}q|q \right)$$

Re-scaling Al-Salam-Chihara polynomials

$Q_n(x) := Q_n(x; a, b|q)$ obtained from the q -analog of the recurrence relation:

$$Q_{n+1}(x) = (2x - (a+b)q^n)Q_n(x) - (1-q^n)(1-abq^{n-1})Q_{n-1}(x)$$

where $Q_0(x) = 1$ and $Q_1(x) = 2x - (a+b)$.

$$L_n(x, y; q) \rightarrow L_n^{(\alpha)}(x) \text{ when } \alpha = 0, y = 1 \text{ and } q \rightarrow 1.$$

q -analog of the generating function?

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- One of the fundamental concepts in combinatorics is enumeration problems, and one of the basic techniques for dealing with the enumeration is **generating functions**.
- A GF f is a representation of a counting function $N : \mathbb{N} \rightarrow \mathbb{N}$ as an element $f(N)$ of some algebra.

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- One of the fundamental concepts in combinatorics is enumeration problems, and one of the basic techniques for dealing with the enumeration is **generating functions**.
- A GF f is a representation of a counting function $N : \mathbb{N} \rightarrow \mathbb{N}$ as an element $f(N)$ of some algebra.
- Several types in specific enumeration problems:
 - Ordinary GF
 - Exponential GF
 - **Eulerian GF**
 - Doubly-exponential GF
 - Chromatic GF
 - Lambert GF

Eulerian generating function

The q -analog of exponential GF $\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$ of the form

$$\sum_{n=0}^{\infty} a_n \frac{t^n}{[n]!_q}$$

is called the **the Eulerian generating functions** for $(a_n)_{n \geq 0}$.

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- q -analog of e^t

$$e_q(t) = \sum_{n \geq 0} \frac{t^n}{[n]!_q}$$

- q -analog of $\ln(1+t)$

$$\ln_q(1+t) = \sum_{n \geq 0} (-1)^{n-1} \frac{t^n}{[n]!_q}$$

II. q -analog of Riordan arrays

- G.-S. Cheon, J.-H. Jung, *Some combinatorial applications of the q -Riordan matrix*, Linear Algebra Appl., 482 (2015), 241-260.
- G.-S. Cheon, J.-H. Jung, Y. Lim, *A q -analogue of the Riordan group*, Linear Algebra Appl., 439 (2013), 4119-4129.

Motivation

- Matrix representation of polynomial sequence

Let $(P_n(x))_{n \geq 0}$ where $P_n(x) = \sum_{k=0}^n p_{n,k} x^k$.

$$\begin{bmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \\ P_n(x) \\ \vdots \end{bmatrix} = \begin{bmatrix} p_{0,0} & & & & \\ p_{1,0} & p_{1,1} & & & 0 \\ p_{2,0} & p_{2,1} & p_{2,2} & & \\ & & \vdots & \ddots & \\ p_{n,0} & p_{n,1} & p_{n,2} & \cdots & p_{n,n} \\ & & \vdots & & \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \\ \vdots \end{bmatrix}.$$

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A $(P_n(x))_{n \geq 0}$ is said to be a **Sheffer for a pair (g, f)** if there exist analytic functions $g(t)$ and $f(t)$ with $g(0) = 1$ and $f(0) = 0$ and $f'(0) = 1$ s.t.

$$\sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} = g(t) e^{xf(t)}.$$

If $(P_n(x))_{n \geq 0}$ is a Sheffer sequence then $L := [p_{n,k}]_{n,k \geq 0}$ can be expressed in terms of $g(t)$ and $f(t)$.

Example of Sheffer sequences

Recall

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) \frac{t^n}{n!} = \frac{1}{(1+t)^{\alpha+1}} e^{x \frac{t}{1+t}}$$

- **Laguerre polynomials** $L_n^{(\alpha)}(x)$ is a Sheffer for $\left(\frac{1}{(1+t)^{\alpha+1}}, \frac{t}{1+t}\right)$.
- **Hermite polynomials** $H_n(x)$ is a Sheffer for $\left(\frac{1}{e^{t^2}}, 2t\right)$.
- **Bernoulli polynomials** $B_n(x)$ is a Sheffer for $\left(\frac{t}{e^t-1}, t\right)$.
- **Euler polynomials** $E_n(x)$ is a Sheffer for $\left(\frac{2}{e^t+1}, t\right)$.

What is a Riordan array?

- Riordan arrays (L. Shapiro et. al, 1991)

$\kappa[[t]]$:= the ring of f.p.s. of the form

$$f(t) = f_0 + f_1 t + f_2 t^2 + \cdots \quad (f_i \in \kappa, \text{ integral domain}).$$

Let $L = [\ell_{n,k}]_{n,k \geq 0}$ be an infinite lower triangular matrix. The L is called a **Riordan array** for (g, f) , denoted $L = (g, f)$ if there exist $g = 1 + g_1 t + g_2 t^2 + \cdots$ and $f = t + f_2 t^2 + \cdots \in \kappa[[t]]$ s.t.

$$g \cdot f^k = \sum_{n \geq k} \ell_{n,k} t^n, \quad k = 0, 1, \dots$$

L is called a **exponential Riordan array** for (g, f) , denoted $L = (g, f)_e$ if

$$g \cdot \frac{f^k}{k!} = \sum_{n \geq k} \ell_{n,k} \frac{t^n}{n!}.$$

Examples

- Pascal matrix

Since $\sum_{n \geq k} \binom{n}{k} t^n = \frac{t^k}{(1-t)^{k+1}} = \frac{1}{1-t} \left(\frac{t}{1-t} \right)^k$,

$$P = \left(\frac{1}{1-t}, \frac{t}{1-t} \right) = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \\ & & \vdots & & \end{bmatrix}$$

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- Stirling matrix of 2nd kind

Since $\sum_{n \geq k} S(n, k) \frac{t^n}{n!} = \frac{(e^t - 1)^k}{k!}$,

$$S = (1, e^t - 1)_e = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 3 & 1 & & \\ 1 & 7 & 6 & 1 & \\ 1 & 15 & 25 & 10 & 1 \\ & & \vdots & & \end{bmatrix}$$

q -analog of Riordan arrays

- Let $\mathcal{E}_q(n)$ be the set of the Eulerian generating functions:

$$a_n \frac{t^n}{[n]!_q} + a_{n+1} \frac{t^{n+1}}{[n+1]!_q} + a_{n+2} \frac{t^{n+2}}{[n+2]!_q} + \cdots, \quad a_n = 1.$$

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- q -calculus
 - q -derivative

$$D_q f(t) = \frac{f(t) - f(qt)}{t - qt}.$$

- k th symbolic power

$$D_q f^{[k]}(t) = [k]_q f^{[k-1]}(t) D_q f(t), \quad f^{[0]}(t) = 1.$$

- q -composition

$$(g \circ_q f)(t) = g[f(t)] = \sum_{n \geq 0} g_n \frac{f^{[n]}(t)}{[n]!_q}, \quad g(t) = \sum_{n \geq 0} g_n \frac{t^n}{[n]!_q}.$$

q -Riordan array

Definition (C & Jung, 2013)

An i.l.t matrix $L = (\ell_{n,k}(q))_{n,k \geq 0}$ is called a **q -Riordan array** denoted $L = (g, f)_q$ if there exist $g \in \mathcal{E}_q(0)$ and $f \in \mathcal{E}_q(1)$ s.t

$$\sum_{n \geq k} \ell_{n,k}(q) \frac{t^n}{n!_q} = g(t) \frac{f^{[k]}(t)}{[k]!_q} \in \mathcal{E}_q(k).$$

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Note that if $q = 0$ then $L = (g, f)_q$ is usual Riordan array, and if $q \rightarrow 1$ then $L = (g, f)_q$ is the exponential Riordan array.

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- $L = (e_q(t), t)_q = \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_q$ is a q -Riordan matrix:

$$\begin{pmatrix} 1 & & & & & \\ 1 & & 1 & & & \\ 1 & & 1+q & & 1 & \\ 1 & & 1+q+q^2 & & 1+q+q^2 & & 1 \\ 1 & 1+q+q^2+q^3 & (1+q+q^2)(1+q^2) & & 1+q+q^2+q^3 & & 1 \\ & & \dots & & & & \end{pmatrix}.$$

Set partitions

- $\Pi_{n,k}$:= the family of all k -partitions $\pi = \{B_1, \dots, B_k\}$ of $\langle n \rangle$ where the elements of each block are written in increasing order and $\min B_1 < \dots < \min B_k$.

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- $\text{inv}(\pi)$:= the number of inversions of π in which all the braces of π are ignored and the elements are treated as a linear array with preserving the order.
- If $\pi = \{\{1, 3\}, \{2, 6\}, \{4, 5\}\} \in \Pi_{6,3}$ then $\text{inv}(\pi) = \text{inv}(132645) = 3$.

Combinatorial interpretation of q -Riordan array

Theorem (C & Jung, 2013)

Let $(G, F)_q = [\alpha_{n,k}(q)]_{n,k \geq 0}$ be a q -Riordan array where $G = \sum_{n \geq 0} g(n) \frac{t^n}{n!_q} \in \mathcal{E}_q(0)$ and $F = \sum_{n \geq 1} f(n) \frac{t^n}{n!_q} \in \mathcal{E}_q(1)$ for some counting functions g and f . Then

$$\alpha_{n,k}(q) = \sum_{\{B_1, \dots, B_{k+1}\} \in \Pi_{n+1, k+1}} g(|B_1| - 1) f(|B_2|) \cdots f(|B_{k+1}|) q^{\text{inv}(\pi)}.$$

Sketch of Proof

- Gessel, 1982

Let $\frac{f^{[k]}(t)}{[k]!_q} = \sum_{n \geq k} c_{n,k} \frac{t^n}{[n]!_q}$ for $f(t) = \sum_{n \geq 1} f_n \frac{t^n}{[n]!_q}$. Then

$$c_{n,k} = \sum_{j=k-1}^{n-1} \begin{bmatrix} n-1 \\ j \end{bmatrix}_q f_{n-j} c_{j,k-1} \quad c_{0,0} = 1, \quad c_{j,0} = 0 \quad (j \geq 1).$$

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- Since $\alpha_{n,k}(q) = \sum_{j=k}^n \begin{bmatrix} n \\ j \end{bmatrix}_q g_{n-j} c_{j,k}$, we obtain recursively:

$$\alpha_{n,k}(q) = \sum_{0 < j_1 < \dots < j_k < n} \left(\begin{bmatrix} n \\ j_k \end{bmatrix}_q \begin{bmatrix} j_k - 1 \\ j_{k-1} \end{bmatrix}_q \dots \begin{bmatrix} j_2 - 1 \\ j_1 \end{bmatrix}_q \right) g(n - j_k) f(j_k - j_{k-1}) \dots f(j_1).$$

- Let $\pi = \{B_1, \dots, B_{k+1}\} \in \Pi_{n+1,k+1}$ where $|B_i| = b_i$. Then

$$\begin{bmatrix} n \\ j_k \end{bmatrix}_q \begin{bmatrix} j_k - 1 \\ j_{k-1} \end{bmatrix}_q \dots \begin{bmatrix} j_2 - 1 \\ j_1 \end{bmatrix}_q = \sum_{\pi \in \Pi_{n+1,k+1}} q^{\text{inv}(\alpha)}.$$

- By some substitution, we obtain desired one.

Example: q-Stirling numbers

Corollary 2.3

If $(1, F)_q = [\beta_{n,k}(q)]_{n,k \geq 0}$ where $F = \sum_{n \geq 1} f(n) \frac{t^n}{n!_q}$ then

$$\beta_{n,k}(q) = \sum_{\pi = \{B_1, \dots, B_k\} \in \Pi_{n,k}} f(|B_1|) \cdots f(|B_k|) q^{\text{inv}(\pi)}$$

with $\beta_{0,0}(q) = 1$ and $\beta_{n,0}(q) = 0$ for $n \geq 1$.

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What happens if $f(n) \equiv 1$ for all $n \geq 1$?

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with $\beta_{0,0}(q) = 1$ and $\beta_{n,0}(q) = 0$ for $n \geq 1$.

What happens if $f(n) \equiv 1$ for all $n \geq 1$?

$$\sum_{\pi \in \Pi_{n,k}} q^{\text{inv}(\pi)} = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q$$

is the q -Stirling numbers of the second kind. Since

$$F = e_q(t) - 1 = \sum_{n \geq 0} \frac{t^n}{n!_q} - 1,$$

$$\sum_{n \geq k} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q \frac{t^n}{n!_q} = \frac{(e_q(t) - 1)^{[k]}}{k!_q}.$$

III. Applications

- G.-S. Cheon, J.-H. Jung, S.-R. Kim, *New q -Laguerre polynomials having factorized permutation interpretation*, J. of Math. Anal. Appl., 470 (2019).
- G.-S. Cheon, J.-H. Jung, *The q -Sheffer sequences of a new type and associated orthogonal polynomials*, Linear Algebra Appl., 491 (2016), 171-186.

q -Sheffer sequence (C & Jung, 2016)

A polynomial sequence $\{s_n(x; q)\}_{n \in \mathbb{N}_0}$ is said to be a q -Sheffer for a pair (g, f) if there exist $g \in \mathcal{E}_q(0)$ and $f \in \mathcal{E}_q(1)$ such that

$$\sum_{n \geq 0} s_n(x; q) \frac{t^n}{[n]_q!} = g e_q[xf]$$

where $e_q(t) = \sum_{n \geq 0} t^n / [n]_q!$.

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where $e_q(t) = \sum_{n \geq 0} t^n / [n]_q!$.

Theorem (C & Jung, 2016)

Let $s_n(x; q) = \sum_{k=0}^n d_{n,k} x^k$. Then $\{s_n(x; q)\}_{n \in \mathbb{N}_0}$ is a q -Sheffer for the pair (g, f) if and only if its coefficient matrix $[d_{n,k}]_{n,k \in \mathbb{N}_0}$ is the q -Riordan array $(g, f)_q$.

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) \frac{t^n}{n!} = \frac{1}{(1+t)^{\alpha+1}} e^{\frac{xt}{1+t}}.$$

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- $\{L_n^{(\alpha)}(x; q)\}_{n \in \mathbb{N}_0}$ is the q -Sheffer for the pair

$$\left(\prod_{j=0}^{\alpha} \frac{1}{1 + q^{j+1}t}, \frac{t}{1+t} \right).$$

Orthogonality

- The first few polynomials:

$$L_0^{(\alpha)}(x; q) = 1,$$

$$L_1^{(\alpha)}(x; q) = x - q[\alpha + 1]_q,$$

$$L_2^{(\alpha)}(x; q) = x^2 - [2]_q[\alpha + 2]_q x + q^2[\alpha + 1]_q[\alpha + 2]_q$$

$$L_3^{(\alpha)}(x; q) = x^3 - (q + [2]_q[\alpha + 2]_q + [\alpha + 4]_q)x^2 \\ + [3]_q[\alpha + 2]_q[\alpha + 3]_q x - q^3[\alpha + 1]_q[\alpha + 2]_q[\alpha + 3]_q.$$

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- $L_n^{(\alpha)}(x; q)$ are orthogonal that satisfy

$$L_{n+1}^{(\alpha)}(x; q) = (x - q[n + \alpha + 1]_q - [n]_q)L_n^{(\alpha)}(x; q) - q[n]_q[n + \alpha]_q L_{n-1}^{(\alpha)}(x; q)$$

where $L_0^{(\alpha)}(x; q) = 1$ and $L_1^{(\alpha)}(x; q) = x - q[\alpha + 1]_q$.

k -factorized permutations

Let $\mathcal{F}_{n,k}$ be the collection of k -factorized permutations $\rho := \rho_1/\rho_2/\cdots/\rho_k$ of $[n]$ where $\min \rho_1 < \cdots < \min \rho_k$.

$$\mathcal{F}_{3,2} = \{1/23, 1/32, 12/3, 21/3, 13/2, 31/2\}.$$

In particular, $\mathcal{F}_{n,1}$ is the collection of permutations of $[n]$.

- An entry of a permutation is called a **right-to-left minimum** if it is smaller than all the entries on its right.
- Let $\mathcal{F}_{n,k}^{(\alpha)} \subset \mathcal{F}_{n+\alpha,k}$ such that $1, 2, \dots, \alpha$ are in the first factor of ρ , and they are right-to-left minima.

$$142/73/65 \in \mathcal{F}_{5,3}^{(2)}$$

- $\text{inv}(142/73/65) = \text{inv}(1427365)$.

Combinatorial interpretation by k -factorized permutations

Theorem 2.3 (C, Jung & Kim, 2019)

Let $L_n^{(\alpha)}(x; q) = \sum_{k=0}^n (-1)^{n-k} a_{n,k}(q) x^k$ be the q -Laguerre polynomials. Then

$$a_{n,k}(q) = \sum_{\rho \in \mathcal{F}_{n,k+1}^{(\alpha)}} q^{\ell_\rho - \alpha + \text{inv}(\rho)}$$

where ℓ_ρ is the length of the first factor in ρ .

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where ℓ_ρ is the length of the first factor in ρ . If $q = 1$ then

$$a_{n,k}(q) = |\mathcal{F}_{n,k+1}^{(\alpha)}| = \frac{n!}{k!} \binom{n+\alpha}{k+\alpha}.$$

Example

Since

$$\mathcal{F}_{2,2}^{(2)} = \{12/34, 12/43, 123/4, \textcolor{blue}{132/4}, 312/4, \textcolor{blue}{124/3}, 142/3, 412/3\},$$

$$\sum_{\rho \in \mathcal{F}_{2,2}^{(2)}} q^{\ell_{\rho} - 2 + \text{inv}(\rho)} = 1 + 2q + \textcolor{blue}{2q^2} + 2q^3 + q^4$$

Thus

$$\begin{aligned} L_2^{(\alpha)}(x; q) &= \sum_{k=0}^2 (-1)^{2-k} \left(\sum_{\rho \in \mathcal{F}_{2,k+1}^{(\alpha)}} q^{\ell_{\rho} - \alpha + \text{inv}(\rho)} \right) x^k \\ &= q^2(1 + 2q + 3q^2 + 3q^3 + 2q^4 + q^5) \\ &\quad - (1 + 2q + 2q^2 + 2q^3 + q^4)x + x^2. \end{aligned}$$

q -rook numbers (Garsia and Remmel, 1986)

- A board is a subset of an $n \times n$ grid of squares. Let $B = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a **Ferrers board** with ℓ building blocks of height $\lambda_1, \lambda_2, \dots, \lambda_\ell$ that satisfies $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_\ell$.
- If $\ell = \lambda_\ell$ then ℓ non-attacking rooks can be placed on the board B . B is called **complete**.
- Two rooks placed in squares (i_1, j_1) and (i_2, j_2) are **noncrossing** if $i_1 < i_2$ and $j_1 < j_2$.
- The **q -rook number** $r_k(B, q)$ for a Ferrers board B is

$$r_k(B, q) = \sum_{C \in \mathcal{C}_k(B)} q^{\text{inv}(C)},$$

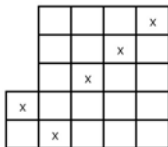
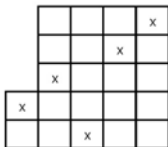
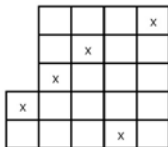
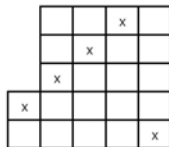
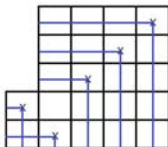
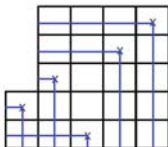
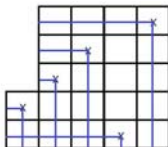
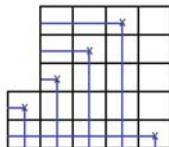
- $\mathcal{C}_k(B)$ is the collection of all placements of k non-attacking rooks on B ,
- $\text{inv}(C)$ is the number of squares of B not crossed out after applying the rule: first cross out all squares which either contain a rook, or are below or to the left of any rook.

				X
			X	
		X		
X				
	X			

				X
			X	
	X			
X				
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			x	
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x				
				x



q -Laguerre polynomial in terms of q -rook numbers

We show that the q -Laguerre polynomial $L_n^{(\alpha)}(x; q)$ can be expressed in terms of q -rook numbers $r_k(B, q)$.

Let $\mathcal{B}_{n,k}^{(\alpha)}$ be the set of complete Ferrers boards $B = (\lambda_1, \dots, \lambda_{n+\alpha})$ of length $n + \alpha$ with k different heights that satisfies $\underbrace{\lambda_{n+\alpha-m+1} = \dots = \lambda_{n+\alpha} = n + \alpha}_{m \text{ times}}$ where

$$\alpha \leq m \leq n + \alpha - k + 1.$$

Let $\mathcal{C}_{n,k}^{(\alpha)}(B)$ denote the collection of all placements of $n + \alpha$ non-attacking rooks on a board $B \in \mathcal{B}_{n,k}^{(\alpha)}$ satisfying the following conditions:

- (i) there are k rooks which have no squares above them;
- (ii) rooks in the first α top rows are pairwise noncrossing.

Complete q -rook number

We call the q -rook number on the complete Ferrers board **complete q -rook number** denoted as $r_{n,h}^{(\alpha)}(B, q)$, i.e.,

$$r_{n,k}^{(\alpha)}(B, q) = \sum_{C \in \mathcal{C}_{n,k}^{(\alpha)}(B)} q^{\text{inv}(C)}.$$

Theorem (C, Jung & Kim, 2019)

Let $L_n^{(\alpha)}(x; q) = \sum_{k=0}^n (-1)^{n-k} a_{n,k}(q) x^k$ be the q -Laguerre polynomial. Then

$$a_{n,k}(q) = \sum_{B \in \mathcal{B}_{n,k+1}^{(\alpha)}} q^{\#(B) - \alpha} r_{n,k+1}^{(\alpha)}(B, q)$$

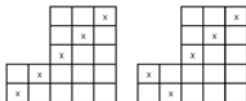
where $\#(B)$ is the number of squares on the top row of the board B .

Example Let $\alpha = 3$, $n = 2$ and $k = 2$. Then $3 \leq m \leq 4$ and

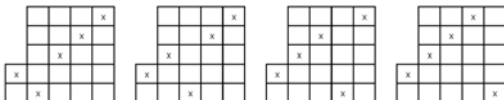
$$\mathcal{B}_{2,2}^{(3)} = \{B_1 = (2, 2, 5, 5, 5), B_2 = (2, 5, 5, 5, 5), B_3 = (1, 5, 5, 5, 5)\}.$$

Thus there are 10 placements:

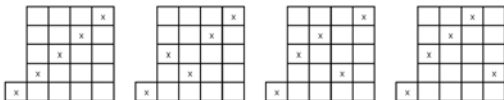
(i) $C_{2,2}^{(3)}(B_1)$



(ii) $C_{2,2}^{(3)}(B_2)$



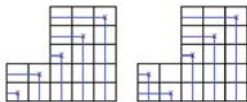
(iii) $C_{2,2}^{(3)}(B_3)$



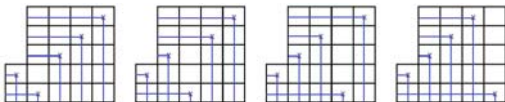
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$$\mathcal{B}_{2,2}^{(3)} = \{B_1 = (2, 2, 5, 5, 5), B_2 = (2, 5, 5, 5, 5), B_3 = (1, 5, 5, 5, 5)\}.$$

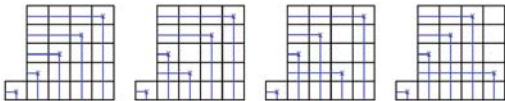
(i) $r_{2,2}^{(3)}(B_1, q) = 1 + q$



(ii) $r_{2,2}^{(3)}(B_2, q) = q + q^2 + q^3 + q^4$



(iii) $r_{2,2}^{(3)}(B_3, q) = 1 + q + q^2 + q^3$



Thus $[x]L_2^{(3)}(x; q) = 1 + 2q + 2q^2 + 2q^3 + 2q^4 + q^5$.

Concluding Remarks

- q -Riordan arrays are very natural q -analog of (exponential) Riordan arrays, and it gives immediately q -analog of Sheffer sequences: such as
 - Hermite polynomials,
 - Bernoulli polynomials,
 - Euler polynomials,
 - Bessel polynomials,
 - $\vdots$

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 - Hermite polynomials,
 - Bernoulli polynomials,
 - Euler polynomials,
 - Bessel polynomials,
 - $\vdots$
- q -Riordan arrays could be used for obtaining new q -analogs of combinatorial objects and their combinatorial interpretations.

Thank you!