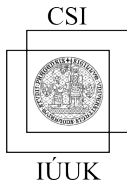


Pattern-Avoiding Fillings of Skew Shapes

Permutation Patterns, 17. 6. 2019

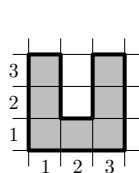
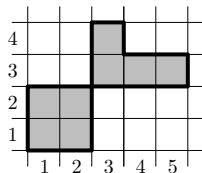
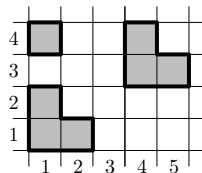
Vít Jelínek Mark Karpilovskij

Computer Science Institute of Charles University in Prague



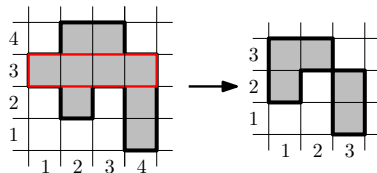
Polyominoes

- A **polyomino** is a finite set of unit squares ('**cells**') in the plane whose vertices lie on the integer grid.
- The **deletion** of a row i in a polyomino P removes all the cells in row i , and moves all cells above row i one step lower. Column deletion is defined similarly.
- A polyomino P is a **subshape** of a polyomino Q if P is obtained from Q by a sequence of deletions of rows and columns.



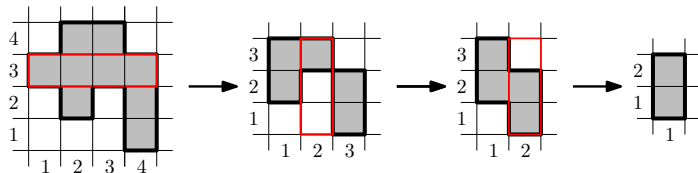
Polyominoes

- A **polyomino** is a finite set of unit squares ('**cells**') in the plane whose vertices lie on the integer grid.
- The **deletion** of a row i in a polyomino P removes all the cells in row i , and moves all cells above row i one step lower. Column deletion is defined similarly.
- A polyomino P is a **subshape** of a polyomino Q if P is obtained from Q by a sequence of deletions of rows and columns.



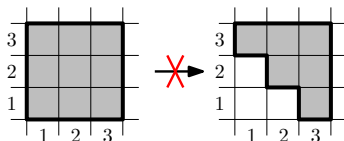
Polyominoes

- A **polyomino** is a finite set of unit squares ('**cells**') in the plane whose vertices lie on the integer grid.
- The **deletion** of a row i in a polyomino P removes all the cells in row i , and moves all cells above row i one step lower. Column deletion is defined similarly.
- A polyomino P is a **subshape** of a polyomino Q if P is obtained from Q by a sequence of deletions of rows and columns.



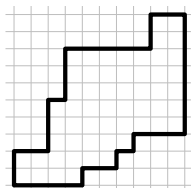
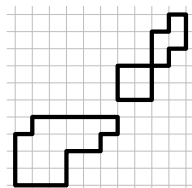
Polyominoes

- A **polyomino** is a finite set of unit squares ('**cells**') in the plane whose vertices lie on the integer grid.
- The **deletion** of a row i in a polyomino P removes all the cells in row i , and moves all cells above row i one step lower. Column deletion is defined similarly.
- A polyomino P is a **subshape** of a polyomino Q if P is obtained from Q by a sequence of deletions of rows and columns.



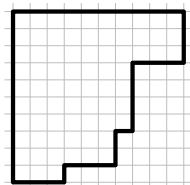
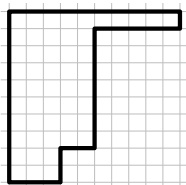
Special polyominoes

- A **skew shape** is formed by cells bounded by two lattice paths with common endpoints, each path consisting only of up-steps and right-steps.
- A **(top-left justified) Ferrers shape**, is a skew shape where the leftmost cell of each row is in the leftmost column, and the topmost cell of each column is in the topmost row.



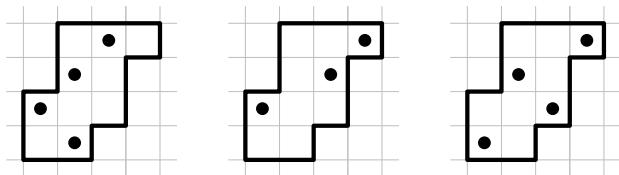
Special polyominoes

- A **skew shape** is formed by cells bounded by two lattice paths with common endpoints, each path consisting only of up-steps and right-steps.
- A **(top-left justified) Ferrers shape**, is a skew shape where the leftmost cell of each row is in the leftmost column, and the topmost cell of each column is in the topmost row.



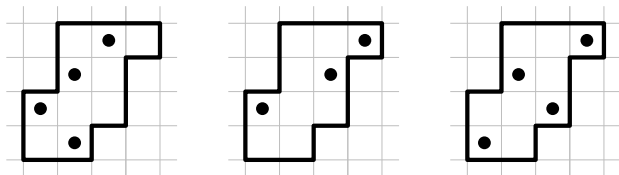
Fillings

- A **filling** of a polyomino assigns to each cell a value 0 or 1.
- A filling is **sparse** if each row and each column has at most one 1-cell.
- A filling is **transversal** if each row and column has exactly one 1-cell.
- ϕ is a **subfilling** of ψ if ϕ is obtained from ψ by a sequence of row deletions, columns deletions, and changes of a 1-cell into a 0-cell.
- Convention: permutation = permutation diagram = transversal filling of a square. The subfilling relation then generalizes classical permutation pattern containment.



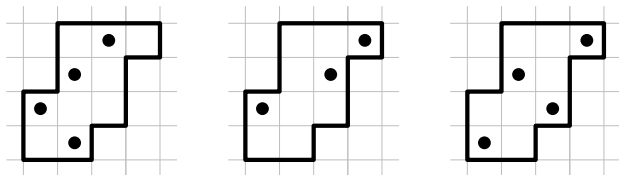
Fillings

- A **filling** of a polyomino assigns to each cell a value 0 or 1.
- A filling is **sparse** if each row and each column has at most one 1-cell.
- A filling is **transversal** if each row and column has exactly one 1-cell.
- ϕ is a **subfilling** of ψ if ϕ is obtained from ψ by a sequence of row deletions, columns deletions, and changes of a 1-cell into a 0-cell.
- Convention: permutation = permutation diagram = transversal filling of a square. The subfilling relation then generalizes classical permutation pattern containment.



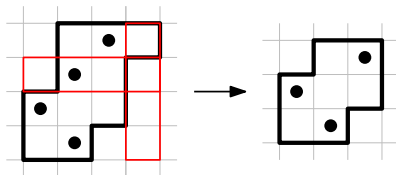
Fillings

- A **filling** of a polyomino assigns to each cell a value 0 or 1.
- A filling is **sparse** if each row and each column has at most one 1-cell.
- A filling is **transversal** if each row and column has exactly one 1-cell.
- ϕ is a **subfilling** of ψ if ϕ is obtained from ψ by a sequence of row deletions, columns deletions, and changes of a 1-cell into a 0-cell.
- Convention: permutation = permutation diagram = transversal filling of a square. The subfilling relation then generalizes classical permutation pattern containment.



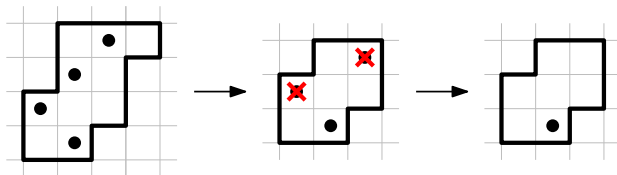
Fillings

- A **filling** of a polyomino assigns to each cell a value 0 or 1.
- A filling is **sparse** if each row and each column has at most one 1-cell.
- A filling is **transversal** if each row and column has exactly one 1-cell.
- ϕ is a **subfilling** of ψ if ϕ is obtained from ψ by a sequence of row deletions, columns deletions, and changes of a 1-cell into a 0-cell.
- Convention: permutation = permutation diagram = transversal filling of a square. The subfilling relation then generalizes classical permutation pattern containment.



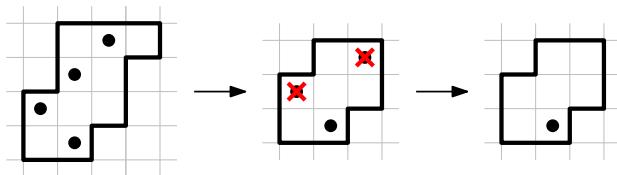
Fillings

- A **filling** of a polyomino assigns to each cell a value 0 or 1.
- A filling is **sparse** if each row and each column has at most one 1-cell.
- A filling is **transversal** if each row and column has exactly one 1-cell.
- ϕ is a **subfilling** of ψ if ϕ is obtained from ψ by a sequence of row deletions, columns deletions, and changes of a 1-cell into a 0-cell.
- Convention: permutation = permutation diagram = transversal filling of a square. The subfilling relation then generalizes classical permutation pattern containment.



Fillings

- A **filling** of a polyomino assigns to each cell a value 0 or 1.
- A filling is **sparse** if each row and each column has at most one 1-cell.
- A filling is **transversal** if each row and column has exactly one 1-cell.
- ϕ is a **subfilling** of ψ if ϕ is obtained from ψ by a sequence of row deletions, columns deletions, and changes of a 1-cell into a 0-cell.
- Convention: permutation = permutation diagram = transversal filling of a square. The subfilling relation then generalizes classical permutation pattern containment.



Wilf equivalences and Wilf orders

For a polyomino P and a filling ϕ , let $\text{Tr}_P(\phi)$ be the number of transversals of P that avoid ϕ .

Let ϕ and ϕ' be two fillings. We write ...

- $\phi \stackrel{S}{\sim} \phi'$ if for every skew shape S , $\text{Tr}_S(\phi) = \text{Tr}_S(\phi')$
(ϕ and ϕ' are skew-equivalent).
- $\phi \stackrel{F}{\sim} \phi'$ if for every (top-left justified) Ferrers shape S ,
 $\text{Tr}_S(\phi) = \text{Tr}_S(\phi')$ (ϕ and ϕ' are Ferrers-equivalent).
- $\phi \stackrel{W}{\sim} \phi'$ if for every square shape S , $\text{Tr}_S(\phi) = \text{Tr}_S(\phi')$
(ϕ and ϕ' are Wilf-equivalent).
- $\phi \stackrel{\leq}{\sim} \phi'$ if for every skew shape S , $\text{Tr}_S(\phi) \leq \text{Tr}_S(\phi')$
(and similarly $\phi \stackrel{\leq}{\sim} \phi'$ and $\phi \stackrel{\geq}{\sim} \phi'$)

Wilf equivalences and Wilf orders

For a polyomino P and a filling ϕ , let $\text{Tr}_P(\phi)$ be the number of transversals of P that avoid ϕ .

Let ϕ and ϕ' be two fillings. We write ...

- $\phi \overset{S}{\sim} \phi'$ if for every skew shape S , $\text{Tr}_S(\phi) = \text{Tr}_S(\phi')$
(ϕ and ϕ' are **skew-equivalent**).
- $\phi \overset{F}{\sim} \phi'$ if for every (top-left justified) Ferrers shape S , $\text{Tr}_S(\phi) = \text{Tr}_S(\phi')$ (ϕ and ϕ' are **Ferrers-equivalent**).
- $\phi \overset{W}{\sim} \phi'$ if for every square shape S , $\text{Tr}_S(\phi) = \text{Tr}_S(\phi')$
(ϕ and ϕ' are **Wilf-equivalent**).
- $\phi \overset{S}{\preceq} \phi'$ if for every skew shape S , $\text{Tr}_S(\phi) \leq \text{Tr}_S(\phi')$
(and similarly $\phi \overset{F}{\preceq} \phi'$ and $\phi \overset{W}{\preceq} \phi'$)

Wilf equivalences and Wilf orders

For a polyomino P and a filling ϕ , let $\text{Tr}_P(\phi)$ be the number of transversals of P that avoid ϕ .

Let ϕ and ϕ' be two fillings. We write ...

- $\phi \overset{S}{\sim} \phi'$ if for every skew shape S , $\text{Tr}_S(\phi) = \text{Tr}_S(\phi')$ (ϕ and ϕ' are **skew-equivalent**).
- $\phi \overset{F}{\sim} \phi'$ if for every (top-left justified) Ferrers shape S , $\text{Tr}_S(\phi) = \text{Tr}_S(\phi')$ (ϕ and ϕ' are **Ferrers-equivalent**).
- $\phi \overset{W}{\sim} \phi'$ if for every square shape S , $\text{Tr}_S(\phi) = \text{Tr}_S(\phi')$ (ϕ and ϕ' are **Wilf-equivalent**).
- $\phi \overset{S}{\preceq} \phi'$ if for every skew shape S , $\text{Tr}_S(\phi) \leq \text{Tr}_S(\phi')$ (and similarly $\phi \overset{F}{\preceq} \phi'$ and $\phi \overset{W}{\preceq} \phi'$)

Wilf equivalences and Wilf orders

For a polyomino P and a filling ϕ , let $\text{Tr}_P(\phi)$ be the number of transversals of P that avoid ϕ .

Let ϕ and ϕ' be two fillings. We write ...

- $\phi \overset{S}{\sim} \phi'$ if for every skew shape S , $\text{Tr}_S(\phi) = \text{Tr}_S(\phi')$ (ϕ and ϕ' are **skew-equivalent**).
- $\phi \overset{F}{\sim} \phi'$ if for every (top-left justified) Ferrers shape S , $\text{Tr}_S(\phi) = \text{Tr}_S(\phi')$ (ϕ and ϕ' are **Ferrers-equivalent**).
- $\phi \overset{W}{\sim} \phi'$ if for every square shape S , $\text{Tr}_S(\phi) = \text{Tr}_S(\phi')$ (ϕ and ϕ' are **Wilf-equivalent**).
- $\phi \overset{S}{\preceq} \phi'$ if for every skew shape S , $\text{Tr}_S(\phi) \leq \text{Tr}_S(\phi')$
(and similarly $\phi \overset{F}{\preceq} \phi'$ and $\phi \overset{W}{\preceq} \phi'$)

Wilf equivalences and Wilf orders

For a polyomino P and a filling ϕ , let $\text{Tr}_P(\phi)$ be the number of transversals of P that avoid ϕ .

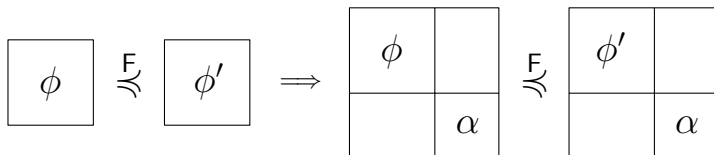
Let ϕ and ϕ' be two fillings. We write ...

- $\phi \overset{S}{\sim} \phi'$ if for every skew shape S , $\text{Tr}_S(\phi) = \text{Tr}_S(\phi')$ (ϕ and ϕ' are **skew-equivalent**).
- $\phi \overset{F}{\sim} \phi'$ if for every (top-left justified) Ferrers shape S , $\text{Tr}_S(\phi) = \text{Tr}_S(\phi')$ (ϕ and ϕ' are **Ferrers-equivalent**).
- $\phi \overset{W}{\sim} \phi'$ if for every square shape S , $\text{Tr}_S(\phi) = \text{Tr}_S(\phi')$ (ϕ and ϕ' are **Wilf-equivalent**).
- $\phi \overset{S}{\preceq} \phi'$ if for every skew shape S , $\text{Tr}_S(\phi) \leq \text{Tr}_S(\phi')$ (and similarly $\phi \overset{F}{\preceq} \phi'$ and $\phi \overset{W}{\preceq} \phi'$)

... but why?

For any permutations $\alpha, \beta, \phi, \phi'$ we have

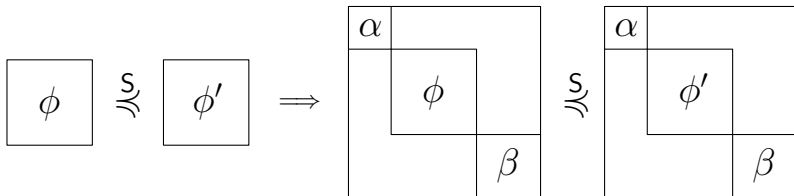
- $\phi \overset{F}{\rightsquigarrow} \phi' \implies \phi \ominus \alpha \overset{F}{\rightsquigarrow} \phi' \ominus \alpha$ (hence $\phi \ominus \alpha \overset{W}{\rightsquigarrow} \phi' \ominus \alpha$)
- $\phi \overset{S}{\rightsquigarrow} \phi' \implies \alpha \ominus \phi \ominus \beta \overset{S}{\rightsquigarrow} \alpha \ominus \phi' \ominus \beta$ (hence $\alpha \ominus \phi \ominus \beta \overset{W}{\rightsquigarrow} \alpha \ominus \phi' \ominus \beta$)
- ... and similarly for $\overset{F}{\sim}, \overset{S}{\sim}$ and $\overset{W}{\sim}$ instead of $\overset{F}{\rightsquigarrow}, \overset{S}{\rightsquigarrow}$ and $\overset{W}{\rightsquigarrow}$.



... but why?

For any permutations $\alpha, \beta, \phi, \phi'$ we have

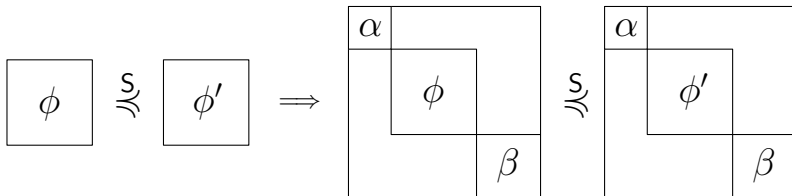
- $\phi \overset{F}{\asymp} \phi' \implies \phi \ominus \alpha \overset{F}{\asymp} \phi' \ominus \alpha$ (hence $\phi \ominus \alpha \overset{W}{\asymp} \phi' \ominus \alpha$)
- $\phi \overset{S}{\asymp} \phi' \implies \alpha \ominus \phi \ominus \beta \overset{S}{\asymp} \alpha \ominus \phi' \ominus \beta$ (hence $\alpha \ominus \phi \ominus \beta \overset{W}{\asymp} \alpha \ominus \phi' \ominus \beta$)
- ... and similarly for $\overset{F}{\sim}, \overset{S}{\sim}$ and $\overset{W}{\sim}$ instead of $\overset{F}{\asymp}, \overset{S}{\asymp}$ and $\overset{W}{\asymp}$.



... but why?

For any permutations $\alpha, \beta, \phi, \phi'$ we have

- $\phi \overset{F}{\asymp} \phi' \implies \phi \ominus \alpha \overset{F}{\asymp} \phi' \ominus \alpha$ (hence $\phi \ominus \alpha \overset{W}{\asymp} \phi' \ominus \alpha$)
- $\phi \overset{s}{\asymp} \phi' \implies \alpha \ominus \phi \ominus \beta \overset{s}{\asymp} \alpha \ominus \phi' \ominus \beta$ (hence $\alpha \ominus \phi \ominus \beta \overset{W}{\asymp} \alpha \ominus \phi' \ominus \beta$)
- ... and similarly for $\overset{F}{\sim}, \overset{s}{\sim}$ and $\overset{W}{\sim}$ instead of $\overset{F}{\asymp}, \overset{s}{\asymp}$ and $\overset{W}{\asymp}$.



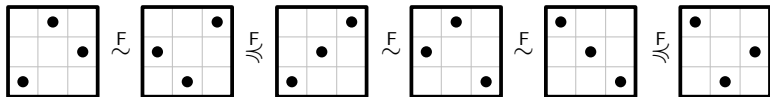
What we know

Ferrers shapes:

- $132 \stackrel{F}{\sim} 213 \stackrel{F}{\asymp} 123 \stackrel{F}{\sim} 321 \stackrel{F}{\sim} 231 \stackrel{F}{\asymp} 312$
- $\iota_k \stackrel{F}{\sim} \delta_k$ where $\iota_k = 123 \cdots k$ and $\delta_k = k(k-1)(k-2) \cdots 1$

Skew shapes:

- $21 \stackrel{S}{\sim} \{12, \phi_{DS}\}$ and hence $21 \stackrel{S}{\asymp} 12$.
- Conjecture: $\delta_k \stackrel{S}{\asymp} \iota_k$ for all k .
- Conjecture: $132 \stackrel{S}{\asymp} 123$.



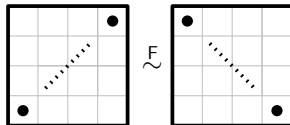
What we know

Ferrers shapes:

- $132 \stackrel{F}{\sim} 213 \stackrel{F}{\asymp} 123 \stackrel{F}{\sim} 321 \stackrel{F}{\sim} 231 \stackrel{F}{\asymp} 312$
- $\iota_k \stackrel{F}{\sim} \delta_k$ where $\iota_k = 123 \cdots k$ and $\delta_k = k(k-1)(k-2) \cdots 1$

Skew shapes:

- $21 \stackrel{S}{\sim} \{12, \phi_{DS}\}$ and hence $21 \stackrel{S}{\asymp} 12$.
- Conjecture: $\delta_k \stackrel{S}{\asymp} \iota_k$ for all k .
- Conjecture: $132 \stackrel{S}{\asymp} 123$.



What we know

Ferrers shapes:

- $132 \stackrel{F}{\sim} 213 \stackrel{F}{\rightsquigarrow} 123 \stackrel{F}{\sim} 321 \stackrel{F}{\sim} 231 \stackrel{F}{\rightsquigarrow} 312$
- $\iota_k \stackrel{F}{\sim} \delta_k$ where $\iota_k = 123 \cdots k$ and $\delta_k = k(k-1)(k-2) \cdots 1$

Skew shapes:

- $21 \stackrel{S}{\sim} \{12, \phi_{DS}\}$ and hence $21 \stackrel{S}{\rightsquigarrow} 12$.
- Conjecture: $\delta_k \stackrel{S}{\rightsquigarrow} \iota_k$ for all k .
- Conjecture: $132 \stackrel{S}{\rightsquigarrow} 123$.

shape DS:



filling ϕ_{DS} :



What we know

Ferrers shapes:

- $132 \stackrel{F}{\sim} 213 \stackrel{F}{\asymp} 123 \stackrel{F}{\sim} 321 \stackrel{F}{\sim} 231 \stackrel{F}{\asymp} 312$
- $\iota_k \stackrel{F}{\sim} \delta_k$ where $\iota_k = 123 \cdots k$ and $\delta_k = k(k-1)(k-2) \cdots 1$

Skew shapes:

- $21 \stackrel{S}{\sim} \{12, \phi_{DS}\}$ and hence $21 \stackrel{S}{\asymp} 12$.
- Conjecture: $\delta_k \stackrel{S}{\asymp} \iota_k$ for all k .
- Conjecture: $132 \stackrel{S}{\asymp} 123$.

$$\begin{array}{|c|c|} \hline \bullet & \\ \hline & \bullet \\ \hline \end{array} \stackrel{S}{\sim} \left\{ \begin{array}{|c|c|} \hline & \bullet \\ \hline \bullet & \\ \hline \end{array}, \begin{array}{|c|c|} \hline \bullet & \\ \hline & \bullet \\ \hline \end{array} \right\}$$

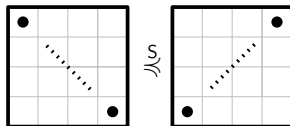
What we know

Ferrers shapes:

- $132 \stackrel{F}{\sim} 213 \stackrel{F}{\asymp} 123 \stackrel{F}{\sim} 321 \stackrel{F}{\sim} 231 \stackrel{F}{\asymp} 312$
- $\iota_k \stackrel{F}{\sim} \delta_k$ where $\iota_k = 123 \cdots k$ and $\delta_k = k(k-1)(k-2) \cdots 1$

Skew shapes:

- $21 \stackrel{S}{\sim} \{12, \phi_{DS}\}$ and hence $21 \stackrel{S}{\asymp} 12$.
- **Conjecture:** $\delta_k \stackrel{S}{\asymp} \iota_k$ for all k .
- Conjecture: $132 \stackrel{S}{\asymp} 123$.



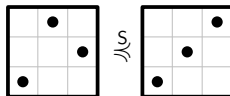
What we know

Ferrers shapes:

- $132 \stackrel{F}{\sim} 213 \stackrel{F}{\asymp} 123 \stackrel{F}{\sim} 321 \stackrel{F}{\sim} 231 \stackrel{F}{\asymp} 312$
- $\iota_k \stackrel{F}{\sim} \delta_k$ where $\iota_k = 123 \cdots k$ and $\delta_k = k(k-1)(k-2) \cdots 1$

Skew shapes:

- $21 \stackrel{S}{\sim} \{12, \phi_{DS}\}$ and hence $21 \stackrel{S}{\asymp} 12$.
- Conjecture: $\delta_k \stackrel{S}{\asymp} \iota_k$ for all k .
- Conjecture: $132 \stackrel{S}{\asymp} 123$.



What is new?

Recall:

- $\phi \stackrel{s}{\sim} \phi' \implies \alpha \ominus \phi \ominus \beta \stackrel{s}{\sim} \alpha \ominus \phi' \ominus \beta.$
- $21 \stackrel{s}{\sim} \{12, \phi_{DS}\}$, i.e., for a skew shape S ,
 $\text{Tr}_S(21) = \text{Tr}_S(\{12, \phi_{DS}\}).$

New results:

- For a skew shape S avoiding DS as subshape and any $k \geq 2$, we have $\text{Tr}_S(\delta_k) = \text{Tr}_S(\iota_k).$
- For any skew shape S , the number of fillings of S that avoid 21 is the same as the number of those that avoid $\{12, \phi_{DS}\}.$
- For any two permutations α, β , there is a finite set of permutations X such that $\alpha \ominus 21 \ominus \beta \stackrel{s}{\sim} \{\alpha \ominus 12 \ominus \beta\} \cup X.$

For $\alpha = \beta = 1$, the last result gives $4321 \stackrel{w}{\sim} \{4231, 5276143\}$,
proved by Burstein & Pantone (2014).

What is new?

Recall:

- $\phi \stackrel{s}{\sim} \phi' \implies \alpha \ominus \phi \ominus \beta \stackrel{s}{\sim} \alpha \ominus \phi' \ominus \beta.$
- $21 \stackrel{s}{\sim} \{12, \phi_{DS}\}$, i.e., for a skew shape S ,
 $\text{Tr}_S(21) = \text{Tr}_S(\{12, \phi_{DS}\}).$

New results:

- For a skew shape S avoiding DS as subshape and any $k \geq 2$, we have $\text{Tr}_S(\delta_k) = \text{Tr}_S(\iota_k).$
- For any skew shape S , the number of fillings of S that avoid 21 is the same as the number of those that avoid $\{12, \phi_{DS}\}.$
- For any two permutations α, β , there is a finite set of permutations X such that $\alpha \ominus 21 \ominus \beta \stackrel{s}{\sim} \{\alpha \ominus 12 \ominus \beta\} \cup X.$

For $\alpha = \beta = 1$, the last result gives $4321 \stackrel{w}{\sim} \{4231, 5276143\}$,
proved by Burstein & Pantone (2014).

What is new?

Recall:

- $\phi \stackrel{s}{\sim} \phi' \implies \alpha \ominus \phi \ominus \beta \stackrel{s}{\sim} \alpha \ominus \phi' \ominus \beta.$
- $21 \stackrel{s}{\sim} \{12, \phi_{DS}\}$, i.e., for a skew shape S ,
 $\text{Tr}_S(21) = \text{Tr}_S(\{12, \phi_{DS}\}).$

New results:

- For a skew shape S avoiding DS as subshape and any $k \geq 2$, we have $\text{Tr}_S(\delta_k) = \text{Tr}_S(\iota_k).$
- For any skew shape S , the number of fillings of S that avoid 21 is the same as the number of those that avoid $\{12, \phi_{DS}\}.$
- For any two permutations α, β , there is a finite set of permutations X such that $\alpha \ominus 21 \ominus \beta \stackrel{s}{\sim} \{\alpha \ominus 12 \ominus \beta\} \cup X.$

For $\alpha = \beta = 1$, the last result gives $4321 \stackrel{w}{\sim} \{4231, 5276143\}$,
proved by Burstein & Pantone (2014).

What is new?

Recall:

- $\phi \stackrel{s}{\sim} \phi' \implies \alpha \ominus \phi \ominus \beta \stackrel{s}{\sim} \alpha \ominus \phi' \ominus \beta.$
- $21 \stackrel{s}{\sim} \{12, \phi_{DS}\}$, i.e., for a skew shape S ,
 $\text{Tr}_S(21) = \text{Tr}_S(\{12, \phi_{DS}\}).$

New results:

- For a skew shape S avoiding DS as subshape and any $k \geq 2$, we have $\text{Tr}_S(\delta_k) = \text{Tr}_S(\iota_k).$
- For any skew shape S , the number of fillings of S that avoid 21 is the same as the number of those that avoid $\{12, \phi_{DS}\}.$
- For any two permutations α, β , there is a finite set of permutations X such that $\alpha \ominus 21 \ominus \beta \stackrel{s}{\sim} \{\alpha \ominus 12 \ominus \beta\} \cup X.$

For $\alpha = \beta = 1$, the last result gives $4321 \stackrel{w}{\sim} \{4231, 5276143\}$,
proved by Burstein & Pantone (2014).

What is new?

Recall:

- $\phi \stackrel{s}{\sim} \phi' \implies \alpha \ominus \phi \ominus \beta \stackrel{s}{\sim} \alpha \ominus \phi' \ominus \beta.$
- $21 \stackrel{s}{\sim} \{12, \phi_{DS}\}$, i.e., for a skew shape S ,
 $\text{Tr}_S(21) = \text{Tr}_S(\{12, \phi_{DS}\}).$

New results:

- For a skew shape S avoiding DS as subshape and any $k \geq 2$, we have $\text{Tr}_S(\delta_k) = \text{Tr}_S(\iota_k).$
- For any skew shape S , the number of fillings of S that avoid 21 is the same as the number of those that avoid $\{12, \phi_{DS}\}.$
- For any two permutations α, β , there is a finite set of permutations X such that $\alpha \ominus 21 \ominus \beta \stackrel{s}{\sim} \{\alpha \ominus 12 \ominus \beta\} \cup X.$

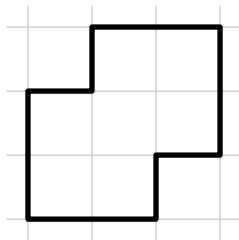
For $\alpha = \beta = 1$, the last result gives $4321 \stackrel{w}{\sim} \{4231, 5276143\}$,
proved by Burstein & Pantone (2014).

Proof ideas – first theorem

Theorem

For a skew shape S avoiding DS as subshape and any $k \geq 2$, we have $\text{Tr}_S(\delta_k) = \text{Tr}_S(\iota_k)$.

DS:

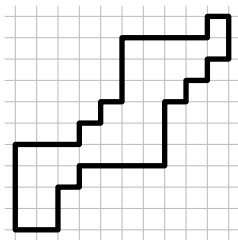


Main tool: Rubey's results on moon polyominoes

Proof ideas – first theorem

Theorem

For a skew shape S avoiding DS as subshape and any $k \geq 2$, we have $\text{Tr}_S(\delta_k) = \text{Tr}_S(\iota_k)$.

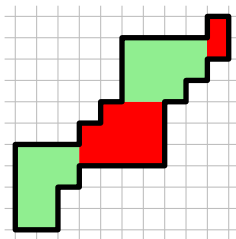


Main tool: Rubey's results on moon polyominoes

Proof ideas – first theorem

Theorem

For a skew shape S avoiding DS as subshape and any $k \geq 2$, we have $\text{Tr}_S(\delta_k) = \text{Tr}_S(\iota_k)$.

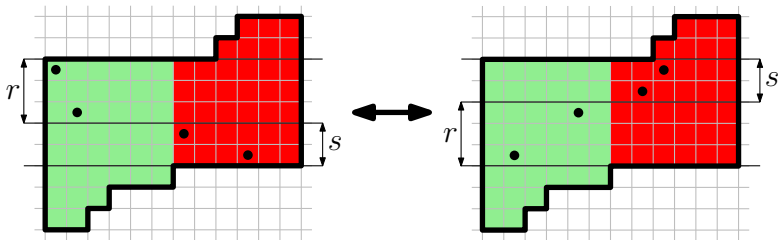


Main tool: Rubey's results on moon polyominoes

Proof ideas – first theorem

Theorem

For a skew shape S avoiding DS as subshape and any $k \geq 2$, we have $\text{Tr}_S(\delta_k) = \text{Tr}_S(\iota_k)$.

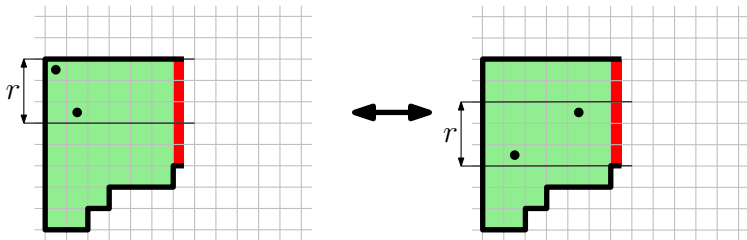


Main tool: Rubey's results on moon polyominoes

Proof ideas – first theorem

Theorem

For a skew shape S avoiding DS as subshape and any $k \geq 2$, we have $\text{Tr}_S(\delta_k) = \text{Tr}_S(\iota_k)$.

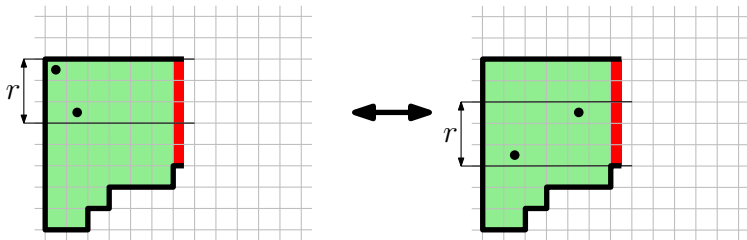


Main tool: Rubey's results on moon polyominoes

Proof ideas – first theorem

Theorem

For a skew shape S avoiding DS as subshape and any $k \geq 2$, we have $\text{Tr}_S(\delta_k) = \text{Tr}_S(\iota_k)$.

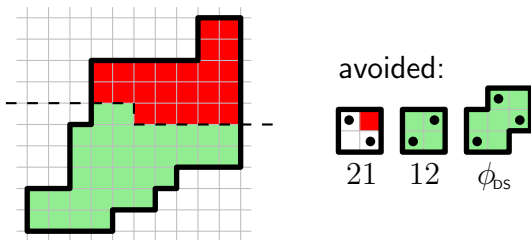


Main tool: Rubey's results on moon polyominoes

Proof ideas – second theorem

Theorem

For any skew shape S , the number of fillings of S that avoid 21 is the same as the number of those that avoid $\{12, \phi_{DS}\}$.

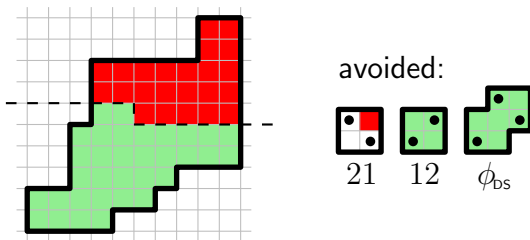


Key idea: the number of fillings is independent from the choice of red/green partition.

Proof ideas – second theorem

Theorem

For any skew shape S , the number of fillings of S that avoid 21 is the same as the number of those that avoid $\{12, \phi_{DS}\}$.



Key idea: the number of fillings is independent from the choice of red/green partition.

Conjectures, further directions

- What more can we say about $\stackrel{w}{\asymp}$ (or even $\stackrel{w}{\sim}$)?
- Conjecture: $\delta_k \stackrel{s}{\asymp} \iota_k$
- For which polyominoes S do we have $\text{Tr}_S(\iota_k) = \text{Tr}_S(\delta_k)$?
- Conjecture: $132 \stackrel{s}{\asymp} 123$
- What about fillings in other (hereditary classes of) polyominoes?
- Wilf-theory of polyominoes?

Conjectures, further directions

- What more can we say about $\stackrel{w}{\asymp}$ (or even $\stackrel{w}{\sim}$)?
- Conjecture: $\delta_k \stackrel{s}{\asymp} \iota_k$
- For which polyominoes S do we have $\text{Tr}_S(\iota_k) = \text{Tr}_S(\delta_k)$?
- Conjecture: $132 \stackrel{s}{\asymp} 123$
- What about fillings in other (hereditary classes of) polyominoes?
- Wilf-theory of polyominoes?

Conjectures, further directions

- What more can we say about $\stackrel{w}{\asymp}$ (or even $\stackrel{w}{\sim}$)?
- Conjecture: $\delta_k \stackrel{s}{\asymp} \iota_k$
- For which polyominoes S do we have $\text{Tr}_S(\iota_k) = \text{Tr}_S(\delta_k)$?
- Conjecture: $132 \stackrel{s}{\asymp} 123$
- What about fillings in other (hereditary classes of) polyominoes?
- Wilf-theory of polyominoes?

Conjectures, further directions

- What more can we say about $\stackrel{w}{\asymp}$ (or even $\stackrel{w}{\sim}$)?
- Conjecture: $\delta_k \stackrel{s}{\asymp} \iota_k$
- For which polyominoes S do we have $\text{Tr}_S(\iota_k) = \text{Tr}_S(\delta_k)$?
- Conjecture: $132 \stackrel{s}{\asymp} 123$
- What about fillings in other (hereditary classes of) polyominoes?
- Wilf-theory of polyominoes?

Conjectures, further directions

- What more can we say about $\stackrel{w}{\asymp}$ (or even $\stackrel{w}{\sim}$)?
- Conjecture: $\delta_k \stackrel{s}{\asymp} \iota_k$
- For which polyominoes S do we have $\text{Tr}_S(\iota_k) = \text{Tr}_S(\delta_k)$?
- Conjecture: $132 \stackrel{s}{\asymp} 123$
- What about fillings in other (hereditary classes of) polyominoes?
- Wilf-theory of polyominoes?

Conjectures, further directions

- What more can we say about $\stackrel{w}{\asymp}$ (or even $\stackrel{w}{\sim}$)?
- Conjecture: $\delta_k \stackrel{s}{\asymp} \iota_k$
- For which polyominoes S do we have $\text{Tr}_S(\iota_k) = \text{Tr}_S(\delta_k)$?
- Conjecture: $132 \stackrel{s}{\asymp} 123$
- What about fillings in other (hereditary classes of) polyominoes?
- Wilf-theory of polyominoes?

The End

Thank you for your attention!