

Hopping from Chebyshev polynomials to permutation statistics

Jordan Tirrell



jtirrell@mtholyoke.edu

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Abstract

We express exponential generating functions counting permutations by the peak number, valley number, double ascent number, and double descent number statistics in terms of the exponential generating function for Chebyshev polynomials. We give a combinatorial proof of this result using monomino-domino tilings, inclusion-exclusion, and the modified Foata–Strehl action (“valley hopping”). We also give a cyclic analogue.

Joint work with Yan Zhuang (Davidson College)

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Eulerian and peak polynomials evaluated at -1

Chebyshev polynomials

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A cyclic analogue

Eulerian polynomials

Definition

For a permutation $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathfrak{S}_n$, we say that $i \in \{1, \dots, n-1\}$ is a *descent* if $\pi_i > \pi_{i+1}$. Define $\text{des}(\pi)$ to be the number of descents of π . We define the *Eulerian polynomials* by

$$A_n(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\text{des}(\pi)}$$

where $\text{des}(\pi)$ denotes the number of descents of π .

These have egf

$$A(t; x) := \sum_{n=1}^{\infty} A_n(t) \frac{x^n}{n!} = \frac{e^{(1-t)x} - 1}{1 - te^{(1-t)x}}$$

Eulerian polynomials evaluated at -1

Fact

Evaluating the Eulerian polynomials at -1 gives a signed version of the tangent numbers.

$$A_n(-1) = \begin{cases} (-1)^{(n-1)/2} E_n, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even,} \end{cases} \quad (1)$$

where E_n is the n th *Euler number* defined by

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sec(x) + \tan(x).$$

These count *alternating permutations* of odd length.

Peak polynomials evaluated at -1

Story time

One lazy day in May 2018, Yan Zhuang was evaluating other permutation statistic polynomials at -1 .

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Definition

For a permutation $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathfrak{S}_n$, we say $i \in \{2, \dots, n-1\}$ is a *peak* if $\pi_{i-1} < \pi_i > \pi_{i+1}$. The *peak polynomials* are defined by

$$P_n^{\text{pk}}(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\text{pk}(\pi)},$$

where $\text{pk}(\pi)$ denotes the number of peaks of π .

These have egf

$$P^{\text{pk}}(t; x) := \sum_{n=1}^{\infty} P_n^{\text{pk}}(t) \frac{x^n}{n!} = \frac{1}{\sqrt{1-t} \coth(x\sqrt{1-t}) - 1}.$$

Peak polynomials evaluated at -1

n	1	2	3	4	5	6	7	8	9	10
$P_n^{\text{pk}}(-1)$	1	2	2	-8	-56	-112	848	9088	25216	-310528

This sequence appears in the OEIS as A006673, and its exponential generating function is the logarithmic derivative of that of the Pell numbers 1, 0, 1, 2, 5, 12, 29....

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Questions

Starting with our combinatorial interpretation of $P_n^{\text{pk}}(t)$,

1. Is there a (cancellation-free) combinatorial interpretation of the sequence $P_n^{\text{pk}}(-1)$?
2. Is there a combinatorial explanation of the fact that the egf for $P_n^{\text{pk}}(-1)$ is the logarithmic derivation of the egf for the Pell numbers?

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Questions

Starting with our combinatorial interpretation of $P_n^{\text{pk}}(t)$,

1. Is there a (cancellation-free) combinatorial interpretation of the sequence $P_n^{\text{pk}}(-1)$? **Open**
2. Is there a combinatorial explanation of the fact that the egf for $P_n^{\text{pk}}(-1)$ is the logarithmic derivation of the egf for the Pell numbers? **Yes!**

Peak polynomials evaluated at -1

Story time

Much of the present work was done during Permutation Patterns 2018 at Dartmouth.

Eulerian and peak polynomials evaluated at -1

Interestingly, the egf for $A_n(-1)$ is the logarithmic derivative of the egf of an even simpler sequence.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
f_n	1	0	1	0	1	0	1	0	1	0	1	0	1
g_n	1	0	1	2	5	12	29	70	169	408	985	2378	5741

$$F(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!} = \cosh(x)$$

$$G(x) = \sum_{n \geq 0} g_n \frac{x^n}{n!} = \frac{1}{2} e^x (2 \cosh(x\sqrt{2}) - \sqrt{2} \sinh(x\sqrt{2}))$$

Theorem

$$(a) \quad A(-1; x) = \frac{\frac{d}{dx} F(x)}{F(x)}$$

$$(b) \quad P^{\text{pk}}(-1; x) = \frac{\frac{d}{dx} G(x)}{G(x)}$$

...with a cyclic analogue for derangements

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$$(c) \quad D(-1; x) = \frac{1}{F(x)}$$

$$(d) \quad D^{\text{cpk}}(-1; x) = \frac{1}{G(x)}$$

where $D(t; x)$ and $D^{\text{cpk}}(t; x)$ are derangement analogues TBD

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These are all straightforward to prove with known generating function formulas. We will give a simple combinatorial story instead.

Generalizing to Chebyshev polynomials

Definition

Define polynomials $U_n(s, t)$ by the recurrence

$$U_n(s, t) = 2tU_{n-1}(s, t) - sU_{n-2}(s, t)$$

for $n \geq 2$ with initial values $U_0(s, t) = 1$ and $U_1(s, t) = 2t$. These are a two-parameter variant of the Chebyshev polynomials of the second kind.

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Then

$$f_n = U_{n-2}(-1, 0) \quad \text{and} \quad g_n = U_{n-2}(-1, 1).$$

n	2	3	4	5	6
f_n	1	0	1	0	1
g_n	1	2	5	12	29
$U_{n-2}(s, t)$	1	$2t$	$4t^2 - s$	$8t^3 - 4st$	$16t^4 - 12st^2 + s^2$

$$V(s, t; x)$$

We will work with the shifted exponential generating function

$$V(s, t; x) := \sum_{n=0}^{\infty} U_n(s, t) \frac{x^{n+2}}{(n+2)!} = \frac{x^2}{2!} + 2t \frac{x^3}{3!} + (4t^2 - s) \frac{x^4}{4!} + \cdots .$$

Note that

$$F(x) = 1 + V(-1, 0; x) \quad \text{and} \quad G(x) = 1 + V(-1, 1; x).$$

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It is known that

$$\sum_{n=0}^{\infty} U_n(s, t) \frac{x^{n+1}}{(n+1)!} = e^{xt} \frac{\sinh(x\sqrt{t^2 - s})}{\sqrt{t^2 - s}},$$

from which it follows that

$$1 - sV(s, t; x) = \cosh(x\sqrt{t^2 - s})e^{tx} - \frac{te^{tx} \sinh(x\sqrt{t^2 - s})}{\sqrt{t^2 - s}}.$$

Permutation statistics

Definition

For a permutation $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathfrak{S}_n$, we adopt the convention that $\pi_0 = \pi_{n+1} = \infty$ and characterize each $i \in \{1, \dots, n\}$ as either

- ▶ a *peak* if $\pi_{i-1} < \pi_i > \pi_{i+1}$,
- ▶ a *valley* if $\pi_{i-1} > \pi_i < \pi_{i+1}$,
- ▶ a *double ascent* if $\pi_{i-1} < \pi_i < \pi_{i+1}$, or
- ▶ a *double descent* if $\pi_{i-1} > \pi_i > \pi_{i+1}$.

We denote the number of valleys, double ascents, and double descents by val , dasc , and ddes , respectively. We also write $\text{dbl} = \text{dasc} + \text{ddes}$.

Notation

Definition

For a list of statistics $\text{st}_1, \text{st}_2, \dots, \text{st}_m$, define

$$P_n^{(\text{st}_1, \text{st}_2, \dots, \text{st}_m)}(t_1, t_2, \dots, t_m) := \sum_{\pi \in \mathfrak{S}_n} t_1^{\text{st}_1(\pi)} t_2^{\text{st}_2(\pi)} \dots t_m^{\text{st}_m(\pi)}.$$

and

$$P^{(\text{st}_1, \text{st}_2, \dots, \text{st}_m)}(t_1, t_2, \dots, t_m; x) := \sum_{n=1}^{\infty} P_n^{(\text{st}_1, \text{st}_2, \dots, \text{st}_m)}(t_1, t_2, \dots, t_m) \frac{x^n}{n!}.$$

The distribution of pk, dbl

Theorem (T., Zhuang 2018)

$$P^{(\text{pk}, \text{dbl})}(s, t; x) = \frac{\frac{\partial}{\partial x} V(s, t; x)}{1 - sV(s, t; x)}$$

Recall

$$V(s, t; x) := \sum_{n=0}^{\infty} U_n(s, t) \frac{x^{n+2}}{(n+2)!} = \frac{x^2}{2!} + 2t \frac{x^3}{3!} + (4t^2 - s) \frac{x^4}{4!} + \dots$$

$$\frac{\partial}{\partial x} V(s, t; x) = e^{xt} \frac{\sinh(x\sqrt{t^2 - s})}{\sqrt{t^2 - s}}$$

$$1 - sV(s, t; x) = \cosh(x\sqrt{t^2 - s})e^{tx} - \frac{te^{tx} \sinh(x\sqrt{t^2 - s})}{\sqrt{t^2 - s}}$$

Using these, we can obtain

$$P^{(\text{pk}, \text{dbl})}(s, t; x) = \frac{1}{\sqrt{t^2 - s} \coth(x\sqrt{t^2 - s}) - t}.$$

Proof

Theorem (T., Zhuang 2018)

$$P^{(\text{pk}, \text{dbl})}(s, t; x) = \frac{\frac{\partial}{\partial x} V(s, t; x)}{1 - sV(s, t; x)}$$

Proof (sketch).

We sketch a combinatorial proof using monomino-domino tilings, inclusion-exclusion, and valley hopping.

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Proof (sketch).

We sketch a combinatorial proof using monomino-domino tilings, inclusion-exclusion, and valley hopping. We want show that

$$P_n^{(\text{pk}, \text{dbl})}(s, t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} s^k \sum_B U_{|B_0|-1}(s, t) U_{|B_1|-2}(s, t) \cdots U_{|B_k|-2}(s, t)$$

where the second sum is over all ordered set partitions B of $[n]$ into blocks $B_0, B_1, \dots, B_k$ such that every block other than B_0 has size at least 2.

Monomono-domino tilings

Recall: $U_n(s, t) = 2tU_{n-1}(s, t) - sU_{n-2}(s, t)$

n	2	3	4	5	6
f_n	1	0	1	0	1
g_n	1	2	5	12	29
$U_{n-2}(s, t)$	1	$2t$	$4t^2 - s$	$8t^3 - 4st$	$16t^4 - 12st^2 + s^2$

Proof (cont.).

The right-hand side counts these set partitions together with:

- ▶ a tiling of an $1 \times (|B_0| - 1)$ rectangle with two types of monominoes (colored red and blue), each weighted t , and one type of domino, each weighted $-s$;
- ▶ for each $1 \leq i \leq k$, a tiling of an $1 \times (|B_i| - 2)$ rectangle with the same types of shapes and weights as above;

and each block (other than B_0) is given an additional weight of s .



Inclusion-Exclusion

Proof (cont.).

Now we define a sign-reversing involution. Swap between the first pair of elements (π_i, π_{i+1}) where there is a domino, or where π_i and π_{i+1} are in separate blocks and $\pi_i > \pi_{i+1}$.

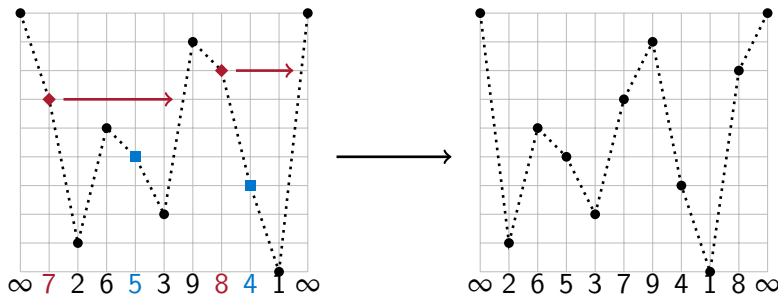


Figure: Example of the sign-reversing involution.

Valley Hopping

Proof (cont.).

Valley-hopping induces a bijection between permutations with no double ascents and a two-coloring on double descents and all permutations.



The quadruple distribution of pk, val, dasc, ddes

The egf for (pk, val, dasc, ddes) then follows from

$$P^{(\text{pk}, \text{val}, \text{dasc}, \text{ddes})}(s, t, u, v; x) = tP^{(\text{pk}, \text{dbl})}\left(st, \frac{1}{2}(u + v); x\right).$$

Theorem (T., Zhuang 2018)

$$P^{(\text{pk}, \text{val}, \text{dasc}, \text{ddes})}(s, t, u, v; x) = \frac{t \frac{\partial}{\partial x} V(st, (u + v)/2; x)}{1 - stV(st, (u + v)/2; x)}$$

A cyclic analogue

A cyclic analogue

Definition

Given $\pi = \pi_1\pi_2 \cdots \pi_n$, we say that π_i is either:

- ▶ a *cyclic peak* if $i < \pi_i > \pi_{\pi_i}$,
- ▶ a *cyclic valley* if $i > \pi_i < \pi_{\pi_i}$,
- ▶ a *cyclic double ascent* if $i < \pi_i < \pi_{\pi_i}$,
- ▶ a *cyclic double descent* if $i > \pi_i > \pi_{\pi_i}$, or
- ▶ a *fixed point* if $i = \pi_i$.

We denote the number of each of these by cpk , cval , cdasc , cddes , and fix respectively. We also write $\text{cdbl} = \text{cdasc} + \text{cddes}$.

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We denote the number of each of these by cpk , cval , cdasc , cddes , and fix respectively. We also write $\text{cdbl} = \text{cdasc} + \text{cddes}$.

Definition

We say that $i \in [n]$ is an *excedance* of π if $i < \pi_i$ and let $\text{exc}(\pi)$ denote the number of excedances of π .

Then i is an excedance of π iff i is a cyclic valley or cyclic double ascent of π , and it is known that the excedance number exc and the descent number des are equidistributed over $\mathfrak{S}_n$.

More Notation

For permutation statistics $\text{st}_1, \text{st}_2, \dots, \text{st}_m$, define

$$D_n^{(\text{st}_1, \text{st}_2, \dots, \text{st}_m)}(t_1, t_2, \dots, t_m) := \sum_{\pi \in \mathfrak{D}_n} t_1^{\text{st}_1(\pi)} t_2^{\text{st}_2(\pi)} \dots t_m^{\text{st}_m(\pi)}$$

and

$$D^{(\text{st}_1, \dots, \text{st}_m)}(t_1, \dots, t_m; x) := 1 + \sum_{n=1}^{\infty} D_n^{(\text{st}_1, \dots, \text{st}_m)}(t_1, \dots, t_m) \frac{x^n}{n!}.$$

where $\mathfrak{D}_n$ denotes the set of *derangements* of n (permutations with no fixed points).

The distribution of cpk, cdbl

Theorem (T., Zhuang 2018)

$$D^{(\text{cpk}, \text{cdbl})}(s, t; x) = \frac{1}{1 - sV(s, t; x)}$$

Proof.

It suffices to show

$$D_n^{(\text{cpk}, \text{cdbl})}(s, t) = \sum_{k=0}^{\lfloor n/2 \rfloor} s^k \sum_B U_{|B_1|-2}(s, t) \cdots U_{|B_k|-2}(s, t)$$

where the second sum is over all ordered set partitions B of $[n]$ into parts $B_1, \dots, B_k$. We interpret the right-hand side as in the first proof (without the initial block B_0 with an ∞) and apply the same sign-reversing involution. The rest of the proof works similarly using cyclic notation and “cyclic valley hopping”, due to Sun and Wang. □

The quadruple derangement distribution

Theorem (T., Zhuang 2018)

$$D^{(\text{cpk}, \text{cval}, \text{cdasc}, \text{cddes})}(s, t, u, v; x) = \frac{1}{1 - stV(st, (u+v)/2; x)}$$

We can use this to obtain the sextuple distribution the sextuple distribution $(\text{cpk}, \text{cval}, \text{cdasc}, \text{cddes}, \text{cyc}, \text{fix})$ over all permutations:

$$\begin{aligned} 1 + P^{(\text{cpk}, \text{cval}, \text{cdasc}, \text{cddes}, \text{cyc}, \text{fix})}(s, t, u, v, w, y; x) \\ = \left(\frac{\alpha e^{(y - \frac{1}{2}(u+v))x}}{\alpha \cosh(\frac{1}{2}\alpha x) - (u+v) \sinh(\frac{1}{2}\alpha x)} \right)^w. \end{aligned}$$

Excedance polynomials

The excedance polynomials $D_n(t) := D_n^{\text{exc}}(t)$ are well-studied.

$$D(t; x) := D^{\text{exc}}(t; x) = \frac{(1-t)e^{-x}}{e^{-(1-t)x} - t}$$

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Corollary

$$D(t; x) = \frac{1}{1 - tV(t, (1+t)/2; x)}$$

Corollary

$$D(-1; x) = F(x)^{-1}$$

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Corollary

$$D(-1; x) = F(x)^{-1}$$

This identity can be used to rederive the classical result due to Roselle that

$$D_n(-1) = \begin{cases} (-1)^{n/2} E_n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Cyclic peak polynomials

Corollary

$$D^{\text{cpk}}(t; x) = \frac{1}{1 - tV(t, 1; x)}$$

$$D^{\text{cpk}}(t; x) = \frac{\sqrt{1-t}e^{-x}}{\sqrt{1-t}\cosh(x\sqrt{1-t}) - \sinh(x\sqrt{1-t})}.$$

n	$D_n^{\text{cpk}}(t)$	n	$D_n^{\text{cpk}}(t)$
1	0	5	$8t + 36t^2$
2	t	6	$16t + 188t^2 + 61t^3$
3	$2t$	7	$32t + 864t^2 + 958t^3$
4	$4t + 5t^2$	8	$64t + 3728t^2 + 9656t^3 + 1385t^4$

Cyclic peak polynomials

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Corollary

$$D^{\text{cpk}}(-1; x) = G(x)^{-1}$$

n	1	2	3	4	5	6	7	8	9	10
$D_n^{\text{cpk}}(-1)$	0	-1	-2	1	28	111	-126	-4067	-26280	53663

Thank you!

Our paper

- ▶ *Hopping from Chebyshev polynomials to permutation statistics*, Jordan O. Tirrell and Yan Zhuang, arXiv:1812.10601 (v1: 27 Dec 2018, v2: 12 Jun 2019).

See also

- ▶ *Hankel Continued fractions and Hankel determinants of the Euler numbers*, Guo-Niu Han, arXiv:1906.00103 (31 May 2019).
- ▶ *On the joint distribution of cyclic valleys and excedances over conjugacy classes of S_n* , M. Crossan Cooper, William S. Jones, and Yan Zhuang, arXiv:1906.05191 (12 Jun 2019).