

Pattern avoidance in permutations and their squares

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Introduction

Definition

A permutation p will be said to be **strongly q -avoiding** if both p and p^2 avoid q .

Notation

Let $\text{Sav}_n(q)$ denote the number of strongly q -avoiding permutations of length n .

The pattern $12 \cdots k$

Theorem

Let k be a positive integer, and let $n \geq (k-1)^3 + 1$. Then $\text{Sav}_n(12 \cdots k) = 0$.

Proof.

Let p be a pattern of length $n \geq (k-1)^3 + 1$ that avoids $12 \cdots k$. Then p is the union of at most $k-1$ decreasing subsequences. Color each subsequence a unique color from $[k-1]$. Consider a decreasing sequence $p_{i_1}, p_{i_2}, \dots, p_{i_m}$ of maximum length $m \geq (k-1)^2 + 1$. Thinking of the $p_{i_1}, p_{i_2}, \dots, p_{i_m}$ as positions, at least k of them must correspond to the same color, say $p_{i_{j_1}}, p_{i_{j_2}}, \dots, p_{i_{j_k}}$ are all positions with the same color. Since $p_{i_{j_1}} > p_{i_{j_2}} > \dots > p_{i_{j_k}}$ and each of these entries correspond to the position of an entry in the same decreasing subsequence of p , we have $p_{i_{j_1}}^2 < p_{i_{j_2}}^2 < \dots < p_{i_{j_k}}^2$. □

Example

Let $k = 3$. Then $n = (k - 1)^3 + 1 = 9$. Consider $p = 895742163$ which avoids 123.

Consider the subsequence $p_1 = 8, p_3 = 5, p_5 = 4, p_6 = 2, p_7 = 1$.

Using these entries as positions, we have

$p_8 = 6, p_5 = 4, p_4 = 7, p_2 = 9, p_1 = 8$.

So $p_1^2 = 6, p_5^2 = 7, p_6^2 = 9$ is a 123 pattern in p .

We have examples up to $k = 6$ to show this bound is sharp.

Example

Let $k = 4$.

Notice p has length $(k - 1)^3 = 27$ and strongly avoids 1234.

$$p = 24\ 21\ 26\ 19\ 23\ 27\ 20\ 25\ 22\ 15\ 12\ 17\ 10\ 14\ 18\ 11\ 16\ 13\ 6\ 3\ 8\ 1\ 5\ 9\ 2\ 7\ 4$$

$$= (1\ 24\ 9\ 22)(2\ 21\ 8\ 25)(3\ 26\ 7\ 20)(4\ 19\ 6\ 27)(5\ 23)(10\ 15\ 18\ 13)(11\ 12)(14)$$

$$p^2 = 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1\ 18\ 17\ 16\ 15\ 14\ 13\ 12\ 11\ 10\ 27\ 26\ 25\ 24\ 23\ 22\ 21\ 20\ 19$$

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$$= (1\ 24\ 9\ 22)(2\ 21\ 8\ 25)(3\ 26\ 7\ 20)(4\ 19\ 6\ 27)(5\ 23)(10\ 15\ 18\ 13)(11\ 12)(14)$$

$$p^2 = 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1 \mid 18\ 17\ 16\ 15\ 14\ 13\ 12\ 11\ 10 \mid 27\ 26\ 25\ 24\ 23\ 22\ 21\ 20\ 19$$

Equivalent patterns

Observation

$$\text{Sav}_q(n) = \text{Sav}_{q^{-1}}(n)$$

for all nonnegative integers n .

Proposition

Let q be any pattern, and let q' denote the reverse complement of q . Then

$$\text{Sav}_q(n) = \text{Sav}_{q'}(n)$$

for all nonnegative integers n .

Corollary

1.

$$\text{Sav}_{312}(n) = \text{Sav}_{231}(n).$$

2.

$$\text{Sav}_{132}(n) = \text{Sav}_{213}(n).$$

The pattern 312

First recall that the grid of a 312-avoiding permutation looks like:



Thus if p strongly avoids 312 and $p_i = 1$, then the permutation $p_1 p_2 \cdots p_{i-1}$ and the reduction of the permutation $p_{i+1} p_{i+2} \cdots p_n$ must each strongly avoid 312.

Therefore, if $\text{Sav}_{312}(z) = \sum_{n \geq 0} \text{Sav}_n(312)z^n$,

and $B(z)$ is the ordinary generating function for the number of strongly 312-avoiding permutations ending in 1,

then we have

$$\text{Sav}_{312}(z) = 1 + B(z)\text{Sav}_{312}(z).$$

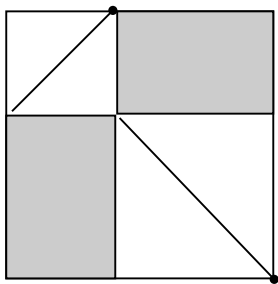
Theorem

For any permutation p ending in 1, the following two statements are equivalent:

(A) The permutation p is strongly 312-avoiding.

(B) The permutation p has form

$$p = (k+1)(k+2) \cdots n \ k(k-1)(k-2) \cdots 1 \text{ where } k \geq \frac{n}{2}.$$



Recall

$$\text{Sav}_{312}(z) = 1 + B(z)\text{Sav}_{312}(z).$$

Rearranging, we get

$$\text{Sav}_{312}(z) = \frac{1}{1 - B(z)}.$$

It follows from our theorem that there are $\lfloor \frac{n}{2} \rfloor$ strongly 312-avoiding permutations of length n and ending in 1 if $n \geq 2$, and there is one such permutation if $n = 1$. Therefore,

$$B(z) = z + \frac{(z+1)z^2}{(1-z^2)^2} = \frac{z^4 - z^3 + z}{(z-1)^2(z+1)}.$$

Hence

$$\text{Sav}_{312}(z) = \frac{-z^3 + z^2 + z - 1}{z^4 - 2z^3 + z^2 + 2z - 1}.$$

The exponential growth rate of the sequence of the numbers $\text{Sav}_{312}(n)$ is about 2.132241882.

The sequence begins: 1, 1, 2, 4, 9, 19, 41, 87, 186, 396, 845, ...

The pattern 321

Call a permutation $p = p_1 p_2 \cdots p_n$ *block-cyclic* if it has both of the following properties:

1. It is possible to cut p into blocks $B_1, B_2, \dots, B_t$ of entries in consecutive positions so that for all $i < j$, the block B_i is on the left of the block B_j , and each entry in B_i is smaller than each entry in B_j .
2. Each block is either a singleton, or its entries can be written in one-line notation as $(a + i) (a + i + 1) \cdots (a + k) (a + 1) \cdots (a + i - 1)$.

That is, each block is a singleton or a power of the cycle $(a + 1 \ a + 2 \cdots a + k)$ that is not the identity.

Example

The following are all block-cyclic (the bars are indicating the border between blocks).

1. $p = 1 \mid 423 \mid 65 \mid 897$,
2. $p = 4123 \mid 5 \mid 8967$,
3. $p = 1 \mid 2 \mid 3 \mid 4 \mid 5$.

Note that every power of a block-cyclic permutation is also block-cyclic and each block-cyclic permutation is 321-avoiding.

Hence block-cyclic permutations are all strongly 321-avoiding.

Let h_n be the number of block-cyclic permutations of length n , and let $H(z) = \sum_{n \geq 0} h_n z^n$. The number of allowed blocks of size k is 1 if $k = 1$, and $k - 1$ if $k > 1$ (since longer blocks cannot be monotone increasing), leading to the formula

$$H(z) = \frac{1}{1 - z - \sum_{k \geq 2} (k - 1)z^k} = \frac{1}{1 - z - \frac{z^2}{(1 - z)^2}} = \frac{(1 - z)^2}{1 - 3z + 2z^2 - z^3}.$$

The exponential growth rate of the sequence h_n is about 2.324717957.

Corollary

The inequality

$$2.3247 \leq \limsup_{n \rightarrow \infty} \sqrt[n]{\text{Sav}_{321}(n)}$$

holds.

Corollary

$\text{Sav}_{321}(n) > \text{Sav}_{312}(n)$ for all n .

The pattern 132

Theorem

(Simion and Schmidt)

The number of involutions that avoid 132 is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Corollary

The growth rate of $\text{Sav}_{132}(n)$ is at least 2.

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Corollary

The growth rate of $\text{Sav}_{132}(n)$ is at least 2.

The sequence $\text{Sav}_{132}(n)$ begins

1, 1, 2, 5, 12, 24, 50, 101, 202, 398, 806, 1568, 3148, 6198, 12306, 24223, ...

Question

Is it true that

$$\lim_{n \rightarrow \infty} \sqrt[n]{\text{Sav}_{132}(n)} = 2?$$

Question

Is it true that for all positive integers n , the inequality

$$\text{Sav}_{132}(n) \leq 2^n$$

holds?

Thank you!