

Catalan words avoiding pairs of length three patterns

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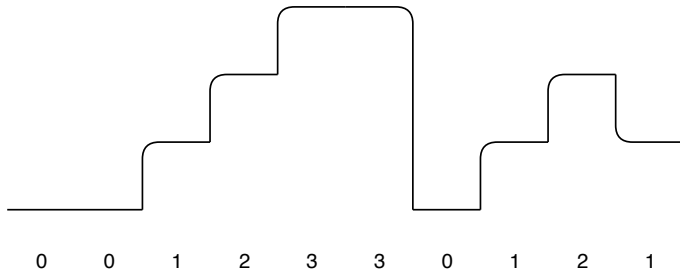
Permutation Patterns 2019, Zurich

Intoduction

Catalan word

- $\mathbf{w} = w_1 w_2 \dots w_n$ is *Catalan word* if $w_1 = 0$ and $0 \leq w_i \leq w_{i-1} + 1$ for $i = 2, 3, \dots, n$.
- $\mathcal{C}_n$ is the set of length n Catalan words.

Example

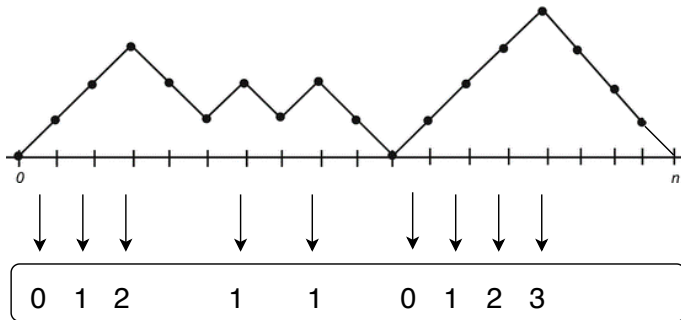


Intoduction

Catalan word vs. Dyck words

There is a direct bijection between the set of Dyck words of semilength n and $\mathcal{C}_n$ the set of Catalan words.

Example



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- $c_n(\pi) = |\mathcal{C}_n(\pi)|$,
- $\mathcal{C}_\pi(x)$ is the generating function for the sequence $(c_n(\pi))_{n \geq 0} = (|\mathcal{C}_n(\pi)|)_{n \geq 0}$.

Example

$\mathcal{C}_4(101) = \{0000, 0001, 0010, 0011, 0012, 0100, 0110, 0111, 0112, 0120, 0121, 0122, 0123\}$

Consider two patterns π and τ of length three, and a fixed n . We look for:

- The number of Catalan words of length n avoiding these two patterns,
- The structural characterization of the set of words in study and their enumerating sequences, using generating functions, recurrence or bijection,
- Recovering known or obtaining new enumerating sequences,
- Giving new interpretations of some integer sequences.

Avoiding a length three pattern

From [B. K. V. 2018]

Pattern π	Sequence $c_n(\pi)$	Generating function	OEIS
012,001,010	2^{n-1}	$\frac{1-x}{1-2x}$	A011782
021	$(n-1) \cdot 2^{n-2} + 1$	$\frac{1-4x+5x^2-x^3}{(1-x)(1-2x)^2}$	A005183
102, 201	$\frac{3^{n-1}+1}{2}$	$\frac{1-3x+x^2}{(1-x)(1-3x)}$	A007051
120, 101	F_{2n-1}	$\frac{1-2x}{1-3x+x^2}$	A001519
011	$\frac{n(n-1)}{2} + 1$	$\frac{1-2x+2x^2}{(1-x)^3}$	A000124
000		$\frac{1-2x^2}{1-x-3x^2+x^3}$	A057960
100	$\lceil \frac{(1+\sqrt{3})^{n+1}}{12} \rceil$	$\frac{1-2x-x^2+x^3}{1-3x+2x^3}$	
110	$\frac{1}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} 2^k - \frac{n-1}{2}$	$\frac{1-3x+2x^2+x^3}{(1-x)^2(1-2x-x^2)}$	
210		$\frac{1-5x+7x^2-x^3-x^4}{(1-2x)(1-4x+3x^2+x^3)}$	

Avoiding a length three and a length two pattern

- $c_n(00, \sigma) = \begin{cases} 0 & \text{if } \sigma = 012, \\ 1 & \text{otherwise.} \end{cases}$

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- $c_n(01, \sigma) = \begin{cases} 0 & \text{if } \sigma = 000, \\ 1 & \text{otherwise.} \end{cases}$
- $c_n(10, \sigma) = \begin{cases} F_{n+1} & \text{if } \sigma = 000, \\ n & \text{if } \sigma \in \{001, 011, 012\}, \\ 2^{n-1} & \text{otherwise.} \end{cases}$

Avoiding two length three patterns

$$C_n(\pi, \tau)$$

Number of Catalan words
avoiding two patterns
of length three

Superfluous patterns

Ultimately constant sequences

Existing counting sequences

New counting sequences

Superfluous patterns

$$\mathcal{C}_n(\pi, \tau) = \mathcal{C}_n(\pi)$$

Example

If $\mathbf{w}$ contains $\tau = 210 \implies \mathbf{w}$ contains $01210 \implies \mathbf{w}$ contains $\pi = 012$

That means, if $\mathbf{w}$ avoids $\pi = 012 \implies \mathbf{w}$ avoids $\tau = 210$

So $\tau = 210$ is superfluous pattern for $\pi = 012$.

Superfluous patterns

Each pattern $\tau \in A$ is a superfluous pattern for π , for

- $A = \{021, 102, 120, 201, 210\}$ and $\pi = 012$,
- $A = \{102, 201, 101\}$ and $\pi = 001$,
- $A = \{021, 102, 120, 201, 210, 100, 101, 110\}$ and $\pi = 010$,
- $A = \{110, 101, 021, 102, 201, 210\}$ and $\pi = 011$,
- $A = \{201, 210\}$ and $\pi = 021$,
- $A = \{102, 201\}$ and $\pi = 101$,
- $A = \{210, 201\}$ and $\pi = 120$,
- $A = \{100\}$ and $\pi = 000$.
- $A = \{210\}$ and $\pi = 110$.

Ultimately constant sequences

Example

$$(c_n(012, 000))_{n \geq 1} = 1, 2, 3, 3, 0, 0, \dots$$

Proof

- $\mathcal{C}_1(012, 000) = \{0\}$
- $\mathcal{C}_2(012, 000) = \{00, 01\}$
- $\mathcal{C}_3(012, 000) = \{001, 011, 010\}$
- $\mathcal{C}_4(012, 000) = \{0011, 0110, 0101\}$
- $\mathcal{C}_5(012, 000) = \emptyset$
- $\mathcal{C}_6(012, 000) = \emptyset$

List of obtained counting sequence

$$c_n(\pi) =$$

- n
- 2^{n-1}
- $\frac{n(n-1)}{2} + 1$
- $(n-1) \cdot 2^{n-2} + 1$
- F_{2n-1}
- F_{n+1}
- $2(n-1)$
- $2^n - n$
- n th Pell number
- $F_n - 1$
- A000125
- A045623
- A116702
- A034943

The case of the counting sequence $\frac{n(n-1)}{2} + 1$

Proof

If w avoids 012 and 100, then:

$$w = \underbrace{\hspace{1.5cm}}_{0^s} \underbrace{\hspace{1.5cm}}_{1^t} \quad s \geq 1, t \geq 0$$

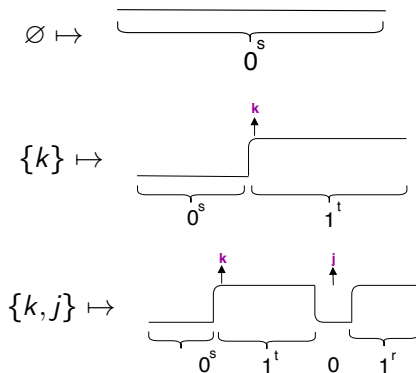
or

$$w = \underbrace{\hspace{1.5cm}}_{0^s} \underbrace{\hspace{1.5cm}}_{1^t} \underbrace{\hspace{1.5cm}}_{0} \underbrace{\hspace{1.5cm}}_{1^r} \quad s \geq 1, t \geq 1 \text{ and } r \geq 0$$

The case of the counting sequence $\frac{n(n-1)}{2} + 1$

Proof

S the family of subsets of $\{2, \dots, n\}$ with at most two element. A bijection $S \rightarrow \mathcal{C}_n(012, 100)$ is



If $\pi = (021, 110)$, then we have

$$\begin{aligned}C_{021,110}(x) &= \frac{(x^5+3x^3-6x^2+4x-1)}{(2x-1)(1-x)^3} \\&= 1 + x + 2x^2 + 5x^3 + 12x^4 + 26x^5 + O(x^6).\end{aligned}$$

New counting sequences

Proof

Let $w \in \mathcal{C}_n(021, 110)$. Then, w has one of the following forms:

$$- \quad w = 0 \mathbf{V} \\ \mathcal{C}_{n-1}^{\cap}(021, 110)$$

$$- \quad w = 0(\underbrace{\mathbf{V}+1}_{\text{w.i.}})$$

$$- \quad w = 01 \underbrace{\mathbf{V}}_{\text{w.i.}}$$

$$- \quad w = \underbrace{\mathbf{V}}_{\text{s.i.}} 0 0..0$$

New counting sequences

Proof

Let $w \in \mathcal{C}_n(021, 110)$. Then, w has one of the following forms:

$$w = \begin{matrix} 0 & \mathbf{V} \\ & \cap \\ & \mathcal{C}_{n-1}(021, 110) \end{matrix} \quad \text{g.f.} \quad x \cdot \mathcal{C}_{(021, 110)}(x),$$

$$w = \begin{matrix} 0(\underbrace{\mathbf{V}+1} \\ \text{w.i.} \end{matrix} \quad \text{g.f.} \quad x \cdot \left(\frac{1-x}{1-2x} - 1 \right),$$

$$w = \begin{matrix} 01 \underbrace{\mathbf{V}} \\ \text{w.i.} \end{matrix} \quad \text{g.f.} \quad x^2 \cdot \left(\frac{1-x}{1-2x} - 1 \right),$$

$$w = \begin{matrix} \underbrace{\mathbf{V}} & 0 & 0..0 \\ \text{s.i.} \end{matrix} \quad \text{g.f.} \quad \frac{x^4}{(1-x)^2}.$$

New counting sequences

With descent,

$$\begin{aligned} \mathcal{C}_{021,110}(x, y) &= \frac{x^5y + x^4y - x^4 - x^3y + 4x^3 - 6x^2 + 4x - 1}{(-1+2x)(1-x)^3} \\ &= 1 + x + 2x^2 + (4+y)x^3 + (8+4y)x^4 + \\ &\quad (16+10y)x^5 + (32+21y)x^6 + O(x^7). \end{aligned}$$

New counting sequences

Proof

Let $w \in \mathcal{C}_n(021, 110)$. Then, w has one of the following forms:

$$w = \begin{matrix} 0 & \mathbf{V} \\ & \cap \\ & \mathcal{C}_{n-1}(021, 110) \end{matrix} \quad \text{g.f.} \quad x \cdot \mathcal{C}_{(021, 110)}(x, y),$$

$$w = \begin{matrix} 0(\underbrace{\mathbf{V}+1} \\ \text{w.i.} \end{matrix} \quad \text{g.f.} \quad x \cdot \left(\frac{1-x}{1-2x} - 1 \right),$$

$$w = \begin{matrix} 01\underbrace{\mathbf{V}} \\ \text{w.i.} \end{matrix} \quad \text{g.f.} \quad y \cdot x^2 \cdot \left(\frac{1-x}{1-2x} - 1 \right),$$

$$w = \begin{matrix} \underbrace{\mathbf{V}} & 0 & 0..0 \\ \text{s.i.} \end{matrix} \quad \text{g.f.} \quad y \cdot \frac{x^4}{(1-x)^2}.$$

Number of Catalan words avoiding 2 patterns of length three

$\pi \setminus \sigma$	000	001	010	100	011	110	101	012	021	102	201	120	210
000	Table 6	F_{n+1}	F_{n+1}	$c_n(000)$	1,2,3,3,3	NEW	2^{n-1}	1,2,3,3,0,0	NEW	NEW	NEW	NEW	NEW
001	-	2^{n-1}	n	$F_n - 1$	n	$\frac{n(n-1)}{2} + 1$	2^{n-1}	n	$\frac{n(n-1)}{2} + 1$	2^{n-1}	2^{n-1}	$\frac{n(n-1)}{2} + 1$	$\binom{n+1}{3} + n + 1$
010	-	-	2^{n-1}	2^{n-1}	n	2^{n-1}	2^{n-1}	n	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}
100	-	-	-	A057960	$2(n+1)$	NEW	Pell number	$\frac{n(n-1)}{2} + 1$	$2^n - n$	NEW	F_{2n-1}	A034943	A267905
011	-	-	-	-	$\frac{n(n-1)}{2} + 1$	$\frac{n(n-1)}{2} + 1$	$\frac{n(n-1)}{2} + 1$	n	$\frac{n(n-1)}{2} + 1$	$\frac{n(n-1)}{2} + 1$	$\frac{n(n-1)}{2} + 1$	$2(n-1)$	$\frac{n(n-1)}{2} + 1$
110	-	-	-	-	-	Table 6	$2^n - n$	$\frac{n(n-1)}{2} + 1$	NEW	A116702	NEW	A034943	NEW
101	-	-	-	-	-	-	F_{2n-1}	$\frac{n(n-1)}{2} + 1$	$2^n - n$	F_{2n-1}	F_{2n-1}	$2^n - n$	$(n-1)2^{n-2} + 1$
012	-	-	-	-	-	-	-	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}
021	-	-	-	-	-	-	-	-	$(n-1)2^{n-2} + 1$	A116702	$(n-1)2^{n-2} + 1$	A045623	$(n-1)2^{n-2} + 1$
102	-	-	-	-	-	-	-	-	-	$\frac{3^{n-1}+1}{2}$	NEW	$(n-1)2^{n-2} + 1$	NEW
201	-	-	-	-	-	-	-	-	-	-	$\frac{3^{n-1}+1}{2}$	F_{2n-1}	NEW
120	-	-	-	-	-	-	-	-	-	-	-	F_{2n-1}	F_{2n-1}
210	-	-	-	-	-	-	-	-	-	-	-	-	Table 6

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001	-	2^{n-1}	n	$F_n - 1$	n	$\frac{n(n-1)}{2} + 1$	2^{n-1}	n	$\frac{n(n-1)}{2} + 1$	2^{n-1}	2^{n-1}	$\frac{n(n-1)}{2} + 1$	$\binom{n+1}{3} + n + 1$
010	-	-	2^{n-1}	2^{n-1}	n	2^{n-1}	2^{n-1}	n	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}
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011	-	-	-	-	$\frac{n(n-1)}{2} + 1$	$\frac{n(n-1)}{2} + 1$	$\frac{n(n-1)}{2} + 1$	n	$\frac{n(n-1)}{2} + 1$	$\frac{n(n-1)}{2} + 1$	$\frac{n(n-1)}{2} + 1$	$2(n-1)$	$\frac{n(n-1)}{2} + 1$
110	-	-	-	-	-	Table 6	$2^n - n$	$\frac{n(n-1)}{2} + 1$	NEW	A116702	NEW	A034943	NEW
101	-	-	-	-	-	-	F_{2n-1}	$\frac{n(n-1)}{2} + 1$	$2^n - n$	F_{2n-1}	F_{2n-1}	$2^n - n$	$(n-1)2^{n-2} + 1$
012	-	-	-	-	-	-	-	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}	2^{n-1}
021	-	-	-	-	-	-	-	-	$(n-1)2^{n-2} + 1$	A116702	$(n-1)2^{n-2} + 1$	A045623	$(n-1)2^{n-2} + 1$
102	-	-	-	-	-	-	-	-	-	$\frac{3^{n-1}+1}{2}$	NEW	$(n-1)2^{n-2} + 1$	NEW
201	-	-	-	-	-	-	-	-	-	-	$\frac{3^{n-1}+1}{2}$	F_{2n-1}	NEW
120	-	-	-	-	-	-	-	-	-	-	-	F_{2n-1}	F_{2n-1}
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010	-	-	2^{n-1}	2^{n-1}	n	2^{n-1}
100	-	-	-	A057960	$2(n+1)$	NEW

- [1] J.-L. Baril, S. Kirgizov and V. Vajnovszki, Descent distribution on Catalan words avoiding a pattern of length at most three.
- [2] A. Baxter and L. Pudwell, Ascent sequences avoiding pairs of patterns, *Electronic Journal of Combinatorics* (electronic), **22**(1), P1.58, 2015.
- [3] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences.