

Widdershins permutations and well-quasi-order

Michael Engen
University of Florida

Joint with Robert Brignall and Vincent Vatter

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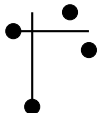
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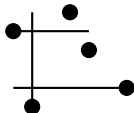
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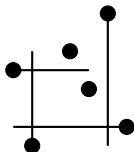
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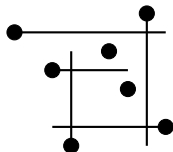
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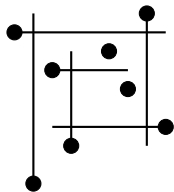
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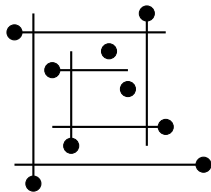
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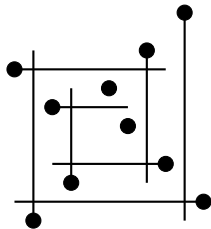
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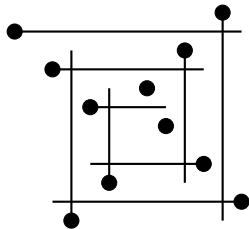
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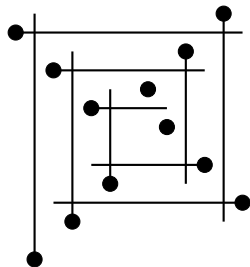
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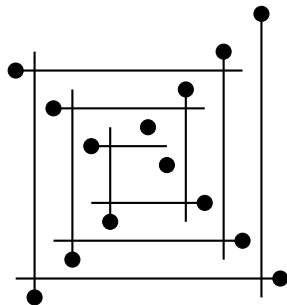
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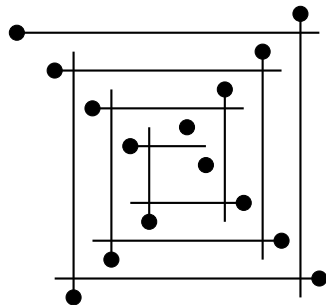
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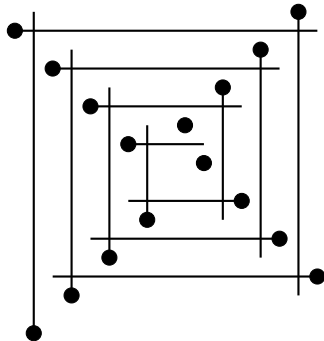
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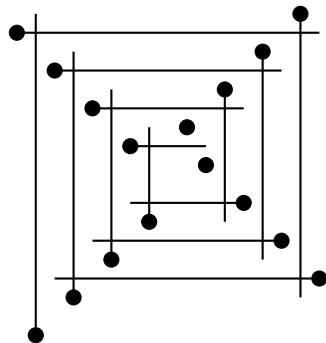
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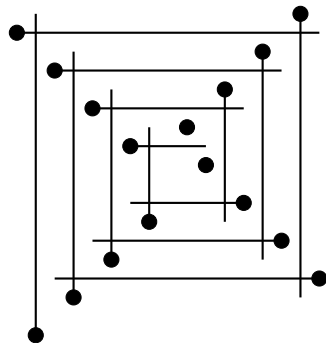


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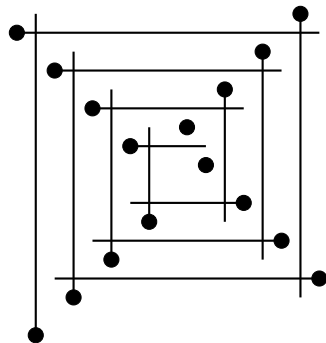
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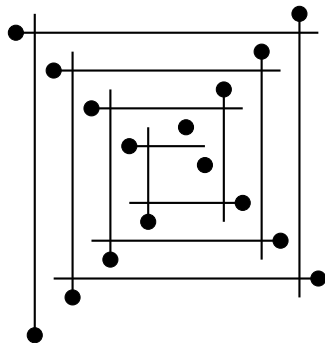
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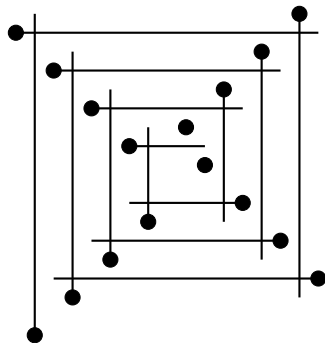
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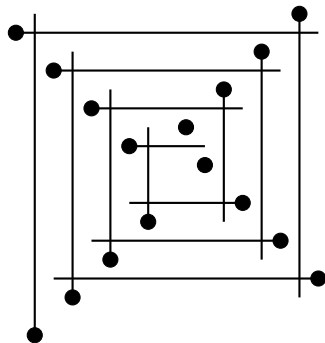
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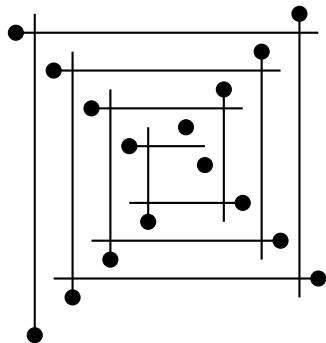
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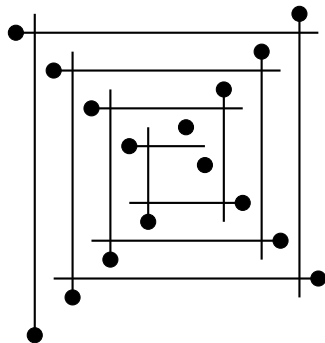
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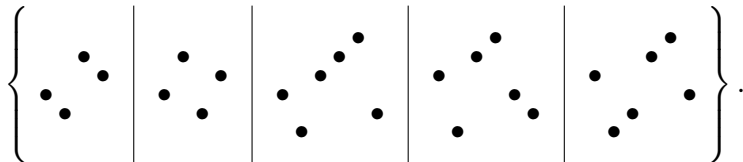
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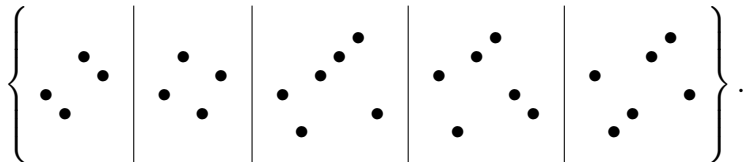
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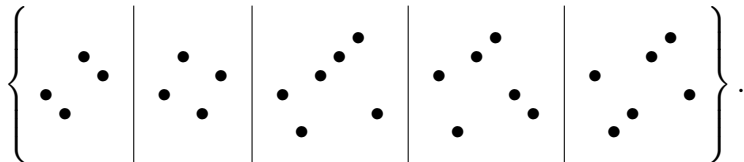


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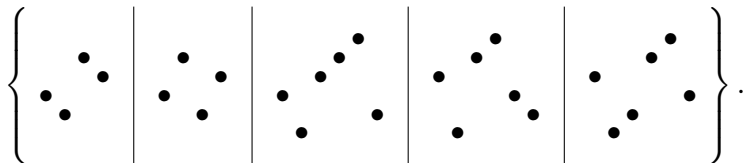
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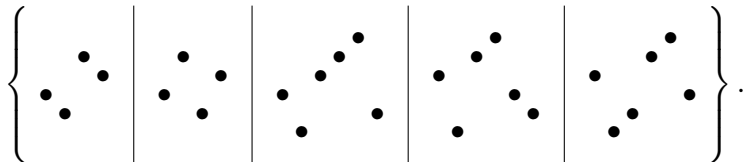
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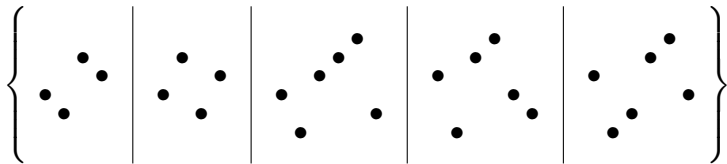
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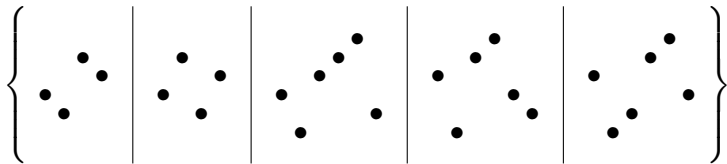
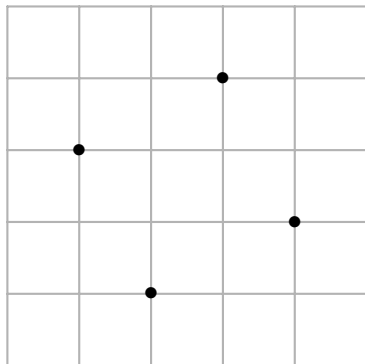
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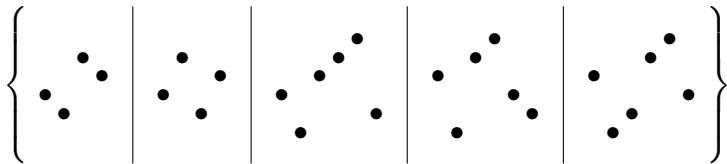
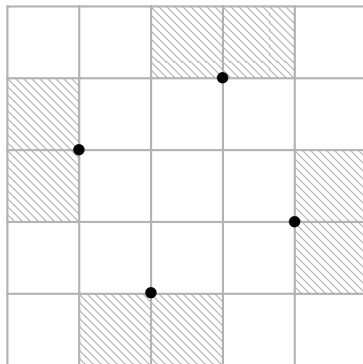
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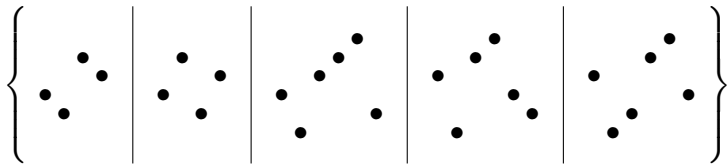
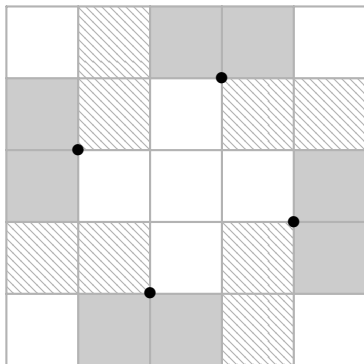
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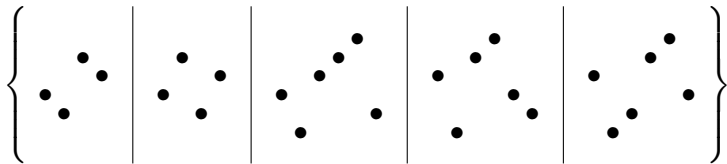
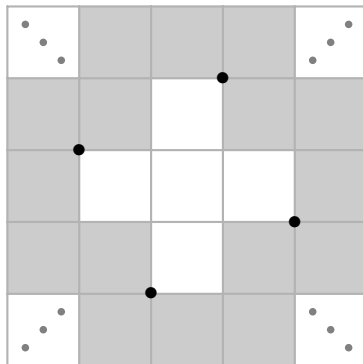
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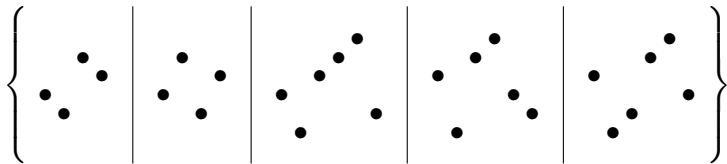
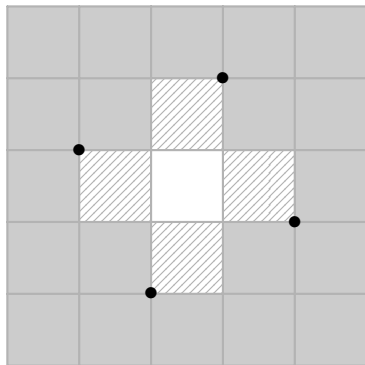
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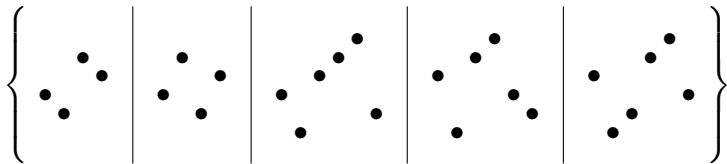
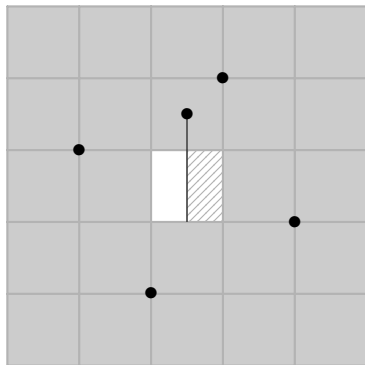
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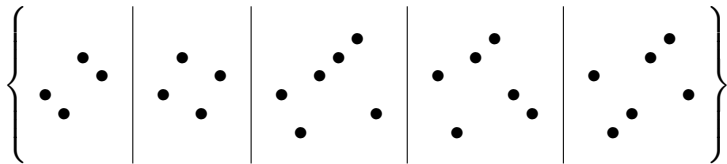
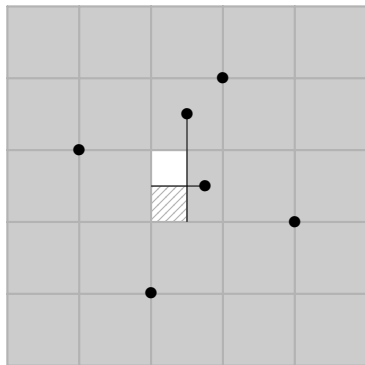
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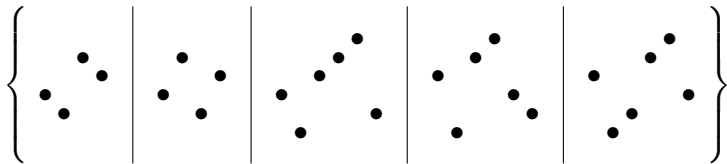
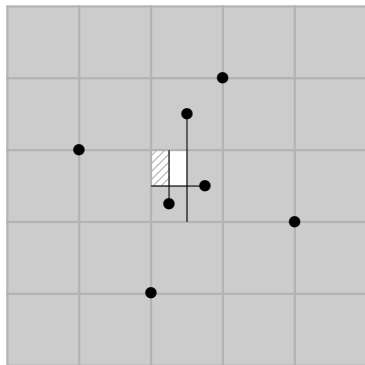
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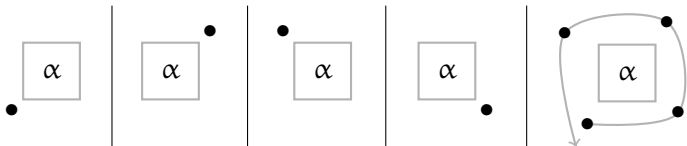


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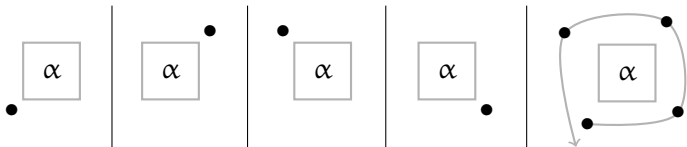
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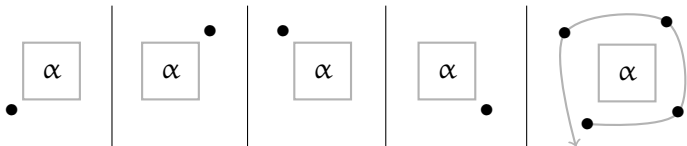
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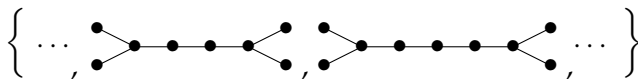
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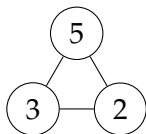
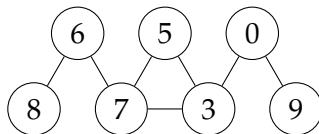
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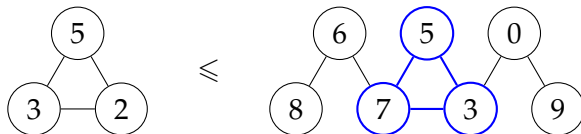
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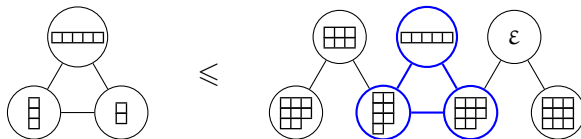
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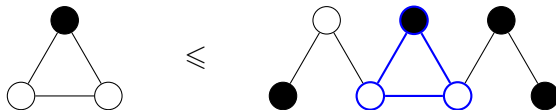
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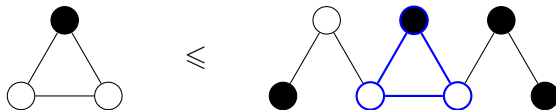
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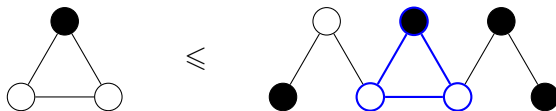
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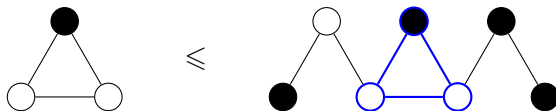
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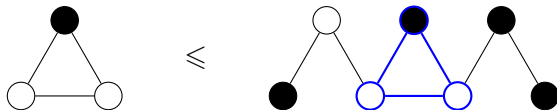
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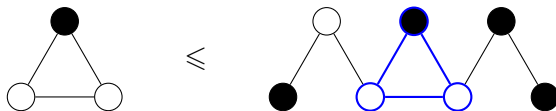
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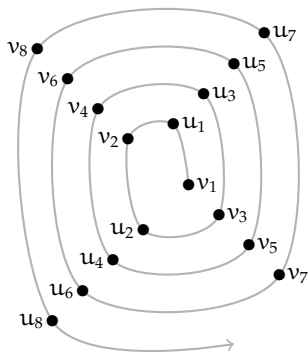
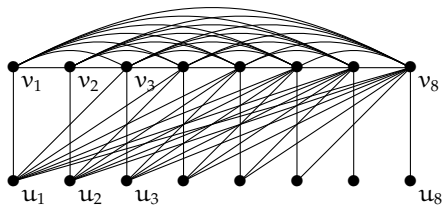
1. $G_{\mathcal{W}}$ is wqo
 - implied by $\mathcal{W}$ being wqo
2. $G_{\mathcal{W}}$ is defined by finitely many minimal forbidden induced subgraphs.
 - $G_{\pi} \cong G_{\pi^{-1}}$, so $G_{\mathcal{W}} = G_{\mathcal{W} \cup \mathcal{W}^{-1}}$
 - $\mathcal{W} \cup \mathcal{W}^{-1}$ is finitely based
 - include those graphs of permutations in the basis of $\mathcal{W} \cup \mathcal{W}^{-1}$ and those which define the split permutation graphs.

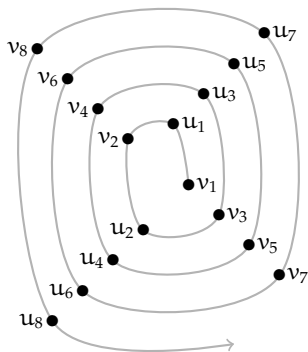
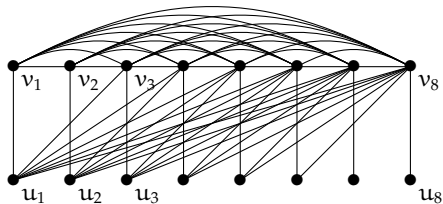
A COUNTEREXAMPLE

Theorem.

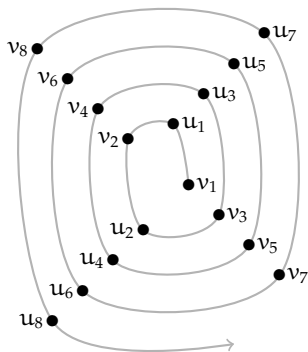
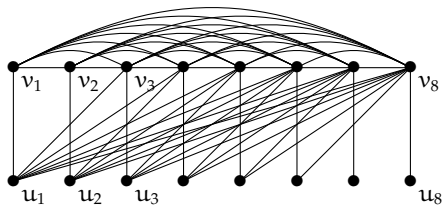
The set of inversion graphs of $\mathcal{W}$, $G_{\mathcal{W}}$, is a counterexample.

1. $G_{\mathcal{W}}$ is wqo
 - implied by $\mathcal{W}$ being wqo
2. $G_{\mathcal{W}}$ is defined by finitely many minimal forbidden induced subgraphs.
 - $G_{\pi} \cong G_{\pi^{-1}}$, so $G_{\mathcal{W}} = G_{\mathcal{W} \cup \mathcal{W}^{-1}}$
 - $\mathcal{W} \cup \mathcal{W}^{-1}$ is finitely based
 - include those graphs of permutations in the basis of $\mathcal{W} \cup \mathcal{W}^{-1}$ and those which define the split permutation graphs.
3. $G_{\mathcal{W}}$ is not lwqo—there exists a wqo set of labels such that the set of all possible labelings of all graphs contains an infinite antichain.

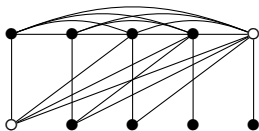
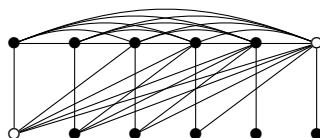




$$\mathbb{L} = \{\bullet, \circ\}$$



$$\mathbb{L} = \{\bullet, \circ\}$$



Thank you.