

Sorting Permutations using Pattern-Avoiding Stacks

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Outline of the Talk

- 1 The γ -avoiding machine.
- 2 Counting classes and non-classes of sortable permutations.
- 3 Two examples of non-classes: the patterns 123 and 132.
- 4 Future directions.

STACK: First-In Last-Out data structure, with two operations:

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Classical Stacksort (Knuth, 1968)

- π is sortable using a single stack $\iff \pi \in \mathcal{AV}(231)$.
- Sortable permutations are enumerated by the Catalan numbers $c_n = \frac{1}{n+1} \binom{2n}{n}$.

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Optimal sorting strategy:

- 1 the stack is increasing $\iff$ the stack avoids 21 (from top to bottom);
- 2 the algorithm is right-greedy.

Two Stacks in Series:

- "2-stack sorting is polynomial", Pierrot, Rossin, 2017.
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Weaker formulations:

-Fix the sorting algorithm.

["Sorting twice with a stack", West, 1993]

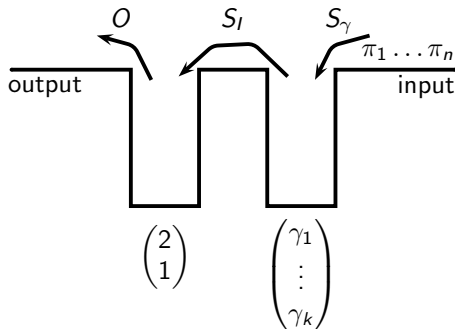
-Impose restrictions on the stacks (decreasing/increasing, popstacks).

["Two stacks in series: a decreasing followed by an increasing stack", Smith, 2014]

Pattern-Avoiding Stacks: the γ -machine.

Rules:

- 1 Two stacks in series, with right-greedy algorithm.
- 2 The first stack avoids γ (γ -stack).
- 3 The second stack avoids 21 (classical stack).



PRIORITY:

- 1 S_γ
- 2 S_I
- 3 O

Some notations:

- $Sort(\gamma) = \{\pi : \pi \text{ is sortable by the } \gamma\text{-machine} \}$.
- $Out^\gamma(\pi) = \text{the output of the } \gamma\text{-stack on input } \pi$.

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MAIN GOALS:

- 1 Characterize $\text{Sort}(\gamma)$
- 2 Enumerate $\text{Sort}_n(\gamma)$

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$\gamma = 21 \longrightarrow$ 2-West sortable permutations.

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$\text{Sort}(21) = \mathcal{AV}(2341, 3\bar{5}241)$ **NOT A CLASS!**

$\gamma = 12 \longrightarrow$ Smith's machine, with greedy algorithm.

$\text{Sort}(12) = \mathcal{AV}(213)$ **CLASS!**

Theorem

$$\pi \in \text{Sort}(12) \iff \pi \in \mathcal{AV}(213).$$

Proof.

[$\Rightarrow$]

- $\pi = L1R \implies \text{Out}^{12}(\pi) = \text{Out}^{12}(L)\text{Out}^{12}(R)1.$

$$\implies \text{If } \pi \geq 213, \text{ then } \text{Out}^\gamma(\pi) \geq 231.$$

[Induction on $|\pi|$]

Theorem

$$\pi \in \text{Sort}(12) \iff \pi \in \mathcal{AV}(213).$$

Proof.

[$\Leftarrow$]

$$\pi \notin \text{Sort}(12) \iff \text{Out}^{12}(\pi) \geq 231.$$

- The 12-stack does not repair inversions
- A 231 in $\text{Out}^{12}(\pi)$ can be traced back to a 213 in π



Enumerating classes and non-classes

How about the enumeration of the patterns γ s.t. $\text{Sort}(\gamma)$ is a class?

Pattern γ	Sortable Permutation	Non-Sortable Pattern
123	4132	132
132	2413	132
213	4132	132
231	361425	1324
312	3142	132
321	$\text{Sort}(321) = \mathcal{AV}(132, 123)$	class

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Conjecture (A. Claesson)

The permutations γ for which $\text{Sort}(\gamma)$ is **not** a permutation class are enumerated by the Catalan numbers.

Hat Lemma

For $\gamma = \gamma_1 \cdots \gamma_k$, define:

- $\hat{\gamma} = \gamma_2 \gamma_1 \gamma_3 \cdots \gamma_k$.
- $\gamma^R = \gamma_k \cdots \gamma_1$.

(i) $\pi \not\geq \gamma^R \implies \text{Out}^\gamma(\pi) = \pi^R$. Then:

$$\pi \in \text{Sort}(\gamma) \iff \pi \not\geq 132$$

Proof. the restriction of the γ -stack is never triggered.

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- $\hat{\gamma} = \gamma_2 \gamma_1 \gamma_3 \cdots \gamma_k$.
- $\gamma^R = \gamma_k \cdots \gamma_1$.

(ii) $\pi \geq \gamma^R \implies \text{Out}^\gamma(\pi) \geq \hat{\gamma} = \gamma_2 \gamma_1 \gamma_3 \gamma_4 \cdots \gamma_k$. Then:

$$\hat{\gamma} \geq 231 \implies \pi \notin \text{Sort}(\gamma)$$

Proof. Take the first occurrence of γ^R : γ_1 pushes γ_2 out and enters above γ_3 .

Hat Lemma

- 1 If $\pi \not\geq \gamma^R$, then $Out^\gamma(\pi) = \pi^R$.
- 2 If $\pi \geq \gamma^R$, then $Out^\gamma(\pi) \geq \hat{\gamma}$.

Corollary (sufficient condition to be a class)

If $\hat{\gamma} \geq 231$, then $Sort(\gamma) = \mathcal{AV}(132, \gamma^R)$.

Proof.

$$Sort(\gamma) \subseteq \mathcal{AV}(132, \gamma^R)$$

- If $\pi \geq \gamma^R$, then $Out^\gamma(\pi) \geq \hat{\gamma} \geq 231$.
- If $\pi \not\geq \gamma^R$, then $\pi \geq 132$ and $Out^\gamma(\pi) = \pi^R \geq 231$.

$$Sort(\gamma) \supseteq \mathcal{AV}(132, \gamma^R)$$

Hat Lemma

- 1 If $\pi \not\geq \gamma^R$, then $\text{Out}^\gamma(\pi) = \pi^R$.
- 2 If $\pi \geq \gamma^R$, then $\text{Out}^\gamma(\pi) \geq \hat{\gamma}$.

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$$\text{Sort}(\gamma) \supseteq \mathcal{AV}(132, \gamma^R)$$

- If $\pi \in \mathcal{AV}(132, \gamma^R)$, then $\text{Out}^\gamma(\pi) = \pi^R$ and $\pi^R \in \mathcal{AV}(231)$.



Lemma α (necessary condition to be a class)

If $\hat{\gamma} \in \mathcal{AV}(231)$, then $\text{Sort}(\gamma)$ is not a permutation class.

Sketch.

- 1 Case by case analysis for $|\gamma| \leq 3$.
- 2 If $|\gamma| \geq 4$, 132 is never sortable: $\text{Out}^\gamma(132) = 231$.
- 3 If $\hat{\gamma} \in \mathcal{AV}(231)$, we construct α :

$$\begin{cases} \alpha \geq 132; \\ \alpha \in \text{Sort}(\gamma). \end{cases}$$

Main Result

Theorem

The following are equivalent:

- 1 $\text{Sort}(\gamma)$ is not a permutation class.
- 2 $\hat{\gamma} \in \mathcal{AV}(231)$.

Corollary

The permutations γ such that $\text{Sort}(\gamma)$ is not a permutation class are enumerated by the Catalan numbers.

Main Result

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The following are equivalent:

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The permutations γ such that $\text{Sort}(\gamma)$ is not a permutation class are enumerated by the Catalan numbers.

MAIN GOAL:

Characterize and enumerate $\text{Sort}(\gamma)$ for particular choices of γ .

HARD CASE: $\gamma = 123$

$\gamma = 123$: $\text{Sort}(123)$ is not a class!

Some enumerative data for $\text{Sort}_n(123)$

- First terms: 1, 2, 5, 13, 35, 99, 295, 920, 2975, 9892, 33605, ... **[A294790]**
- Subtract n from partial sums of partial sums of Catalan numbers.

How to deal with non-classes of sortable permutations?

- Find relevant statistics (link with other combinatorial objects?).
- Find a structural description.

Characterization of $\text{Sort}(123)$

1 $\pi \in \text{Sort}(123) \implies \pi_2 \leq \pi_1 + 1.$

[No Large Ascents]

2 $\pi \in \text{Sort}(123) \iff \pi[id_k, 1, \dots, 1] \in \text{Sort}(123).$

[Consecutive Ascents]

Characterization of $\text{Sort}(123)$

① $\pi \in \text{Sort}(123) \implies \pi_2 \leq \pi_1 + 1.$ [No Large Ascents]

② $\pi \in \text{Sort}(123) \iff \pi[id_k, 1, \dots, 1] \in \text{Sort}(123).$ [Consecutive Ascents]

$\implies$ We can focus on:

$$\text{Sort}^D(123) = \{\pi \in \text{Sort}(123) : \pi_1 > \pi_2\}.$$

③ $\{\pi \in \text{Sort}_n^D(123) : \pi_1 = n\} = \{\pi \in \mathcal{AV}(213) : \pi_1 = n\}.$ [$\pi_1 = n$]

④ Bijection $\varphi: \text{Sort}_n^D(123) \longleftrightarrow \{\pi \in \text{Sort}_{n+1}^D(123) : \pi_1 \neq n+1\}.$ [$\pi_1 \neq n$]

φ is constructive:

The position of the new maximum is uniquely determined.

Characterization of $\text{Sort}(123)$

Each $\pi \in \text{Sort}_n(123)$, $\pi \neq \text{id}$, is obtained as follows:

- 1 Choose $\alpha \in \mathcal{AV}_k(213)$ with $\alpha_1 = k$. [Catalan-type middle part]
- 2 Insert some new maxima according to φ . [Final tail]
- 3 Inflate the first element. [Starting tail]

Corollary

$$f_n^{123} = 1 + \sum_{k=2}^n c_{k-1} \cdot (n - k + 1).$$



HARDER CASE: $\gamma = 132$

The pattern $\gamma = 132$

- Conjectured enumeration: $f_n^{132} = \sum_{k=0}^{n-1} \binom{n-1}{k} c_k$

[A007317]

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- Conjectured enumeration: $f_n^{132} = \sum_{k=0}^{n-1} \binom{n-1}{k} c_k$ [A007317]
- IDEA:** Look at the LTR-minima!

Lemma

$\pi = m_1 B_1 m_2 B_2 \cdots m_k B_k$, with $m_1, \dots, m_k$ LTR-minima of π .

$$\implies \text{Out}(\pi) = \tilde{B}_1 \tilde{B}_2 \cdots \tilde{B}_k m_k \cdots m_2 m_1$$

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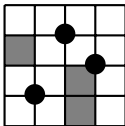
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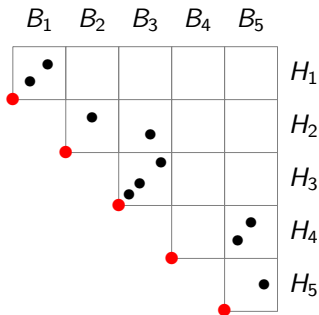
Corollary

$\pi \in \text{Sort}(132) \iff \pi$ avoids 2314 and $(132, \{(0, 2), (2, 0), (2, 1)\})$.



The grid-decomposition of

$\pi = \mathbf{13} \ 14 \ 15 \ \mathbf{10} \ 12 \ \mathbf{6} \ 7 \ 8 \ 11 \ 9 \ \mathbf{3} \ \mathbf{1} \ 4 \ 5 \ 2.$



How to place a new rightmost element?

❶ Unique spot for each legal box.

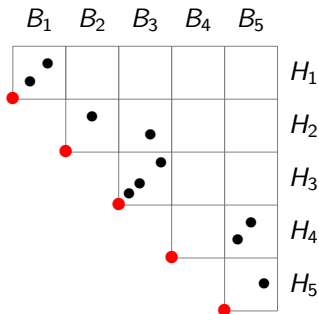
$\implies$ Set partition!

❷ Some boxes are not legal.

$\implies$ Pattern-Avoidance!

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How to read the set partition from the picture?

Read from left to right and record the index of the horizontal strip!

$\Rightarrow$ 1 1 1 2 2 3 3 3 2 3 4 5 4 4 5

[Restricted Growth Function]

Theorem

There is a bijection between:

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Corollary [V. Jelínek, T. Mansour]

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Open Problems:

- 1 Derive the formula directly
- 2 If we refine with the number of LTR-minima:

$$f_n^{132}(j \text{ blocks}) = \sum_{k=j}^{n-1} \binom{n-1}{k} n(k, j)$$

Future work

- Three patterns of length 3 are still to be solved.
- Wilf-classes for classes of sortable permutations.
- Develop a more general theory for the γ -machines: what about the non-classes?

~ Theorem



Let $\mathcal{M} =$ Then:

$$\text{Sort}(\gamma) \subseteq \mathcal{AV}(\mathcal{M}) \iff \gamma \neq 12 \ominus \alpha, \text{ with } \alpha \in \mathcal{AV}(231)$$