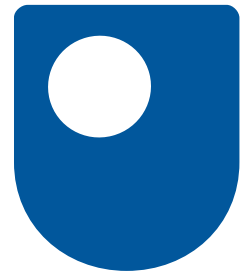

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The growth of the Möbius function on the permutation poset

David Marchant

Permutation Patterns 2019

Universität Zürich (UZH), Switzerland.

18th June 2019

The Möbius function

The *(principal) Möbius function* $\mu[\pi]$ is defined on the permutation poset as

$$\mu[\pi] = \begin{cases} 1 & \text{if } \pi = 1, \\ -\sum_{\lambda < \pi} \mu[\lambda] & \text{otherwise} \end{cases}$$

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Hall's Theorem:

$$\mu[\pi] = \sum_{c \in \mathcal{C}(\pi)} (-1)^{|c|}$$

where $\mathcal{C}(\pi)$ is the set of chains between 1 and π.

What do we mean by the growth of $\mu[\pi]$?

Let $\max_\mu(n)$ be the largest absolute value of $\mu[\pi]$ when π is length n.

Then "growth" attempts to place limits on $\max_\mu(n)$ as a function of n.

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Today:

- Growth is at least exponential.

2413-balloons

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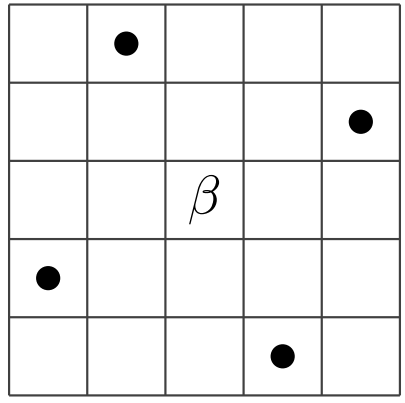

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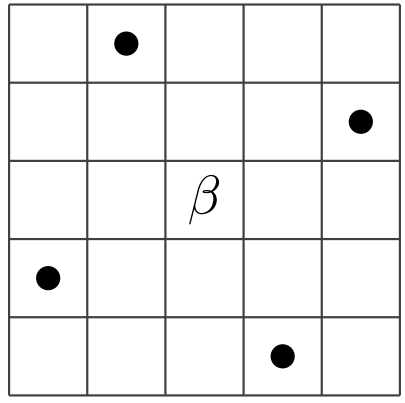

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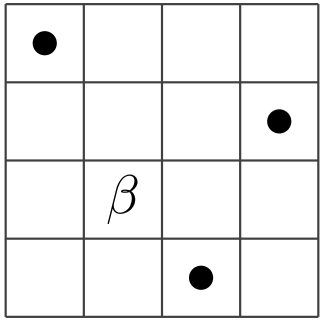
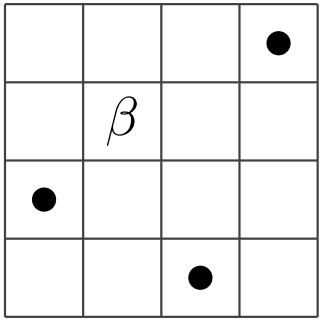
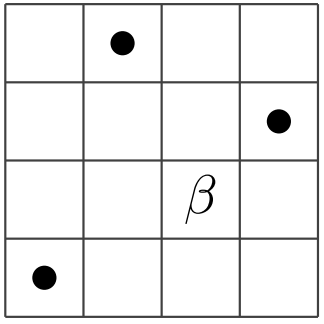
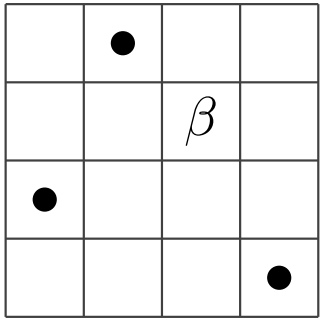
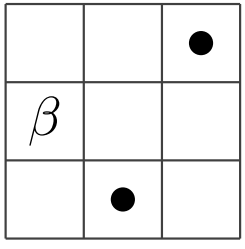
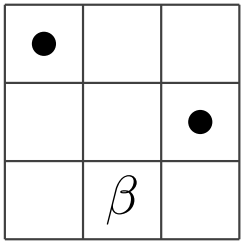
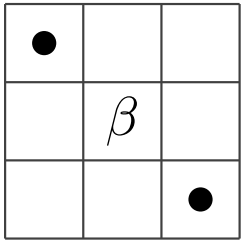
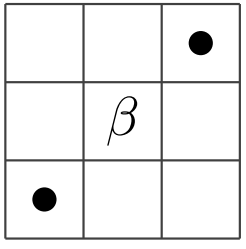
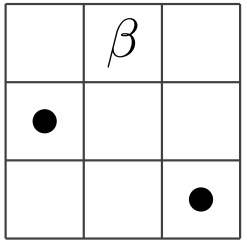
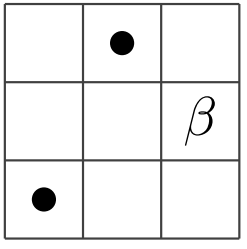
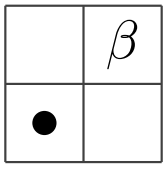
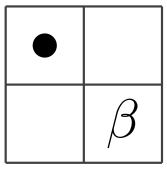
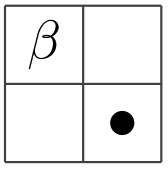
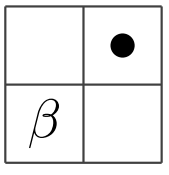
This is equivalent to the inflation $25314[1, 1, \beta, 1, 1]$.

Reductions of a 2413-balloon

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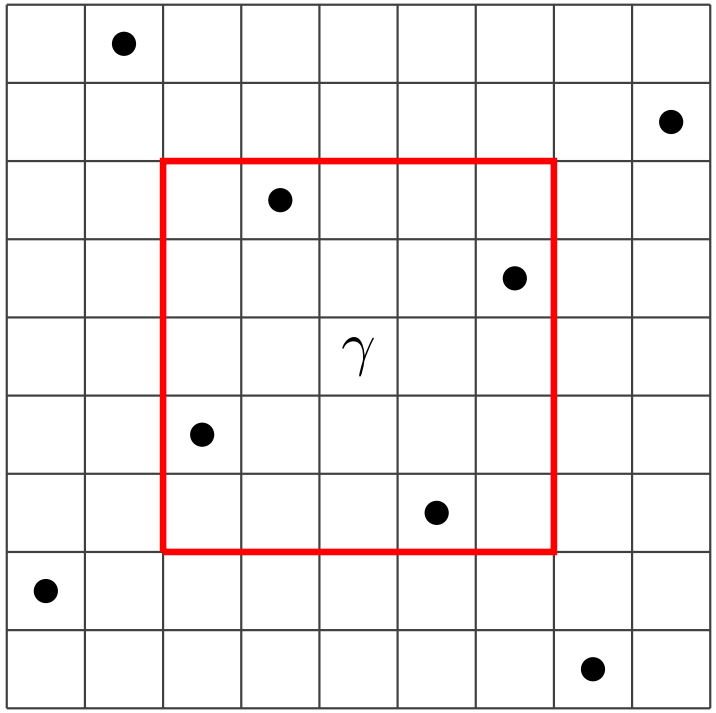

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So by Hall's Theorem

$$\mu[\pi] = \sum_{c \in \mathcal{R}} (-1)^{|c|} + \sum_{c \in \mathcal{G}} (-1)^{|c|} + \sum_{c \in \mathcal{B}} (-1)^{|c|}$$

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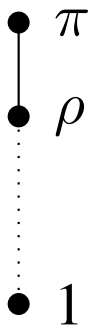
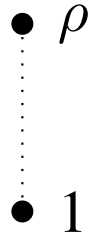


Chain in $[1, \rho]$



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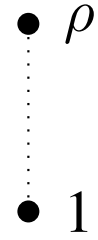
Chain in $\mathcal{R}_\rho$

Chain in $[1, \rho]$



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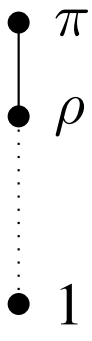

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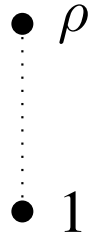

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Finding the growth

So if Red is the set of permutations that are reductions of π, then

$$\begin{aligned}\mu[\pi] &= \sum_{c \in \mathcal{R}} (-1)^{|c|} \\ &= -\sum_{\lambda \in \mathsf{Red}} \mu[\lambda]\end{aligned}$$

Handling the reductions

Theorem (Burstein, Jelínek, Jelínková and Steingrímsson (2011))

If $\tau \neq 1 \oplus \gamma$, *then* $\mu[1 \oplus \tau] = -\mu[\tau]$.

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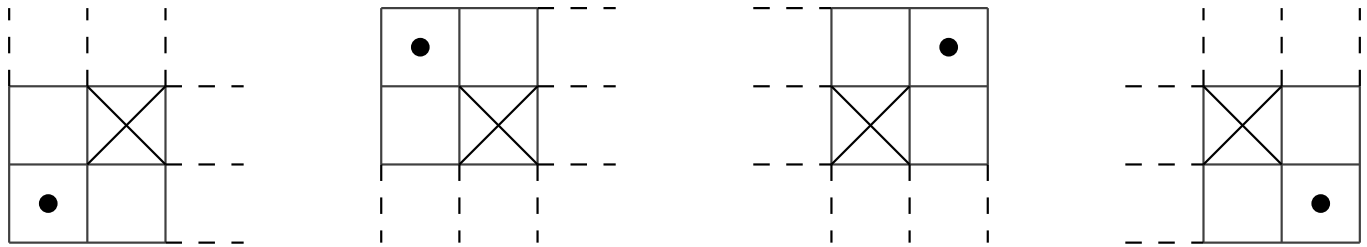

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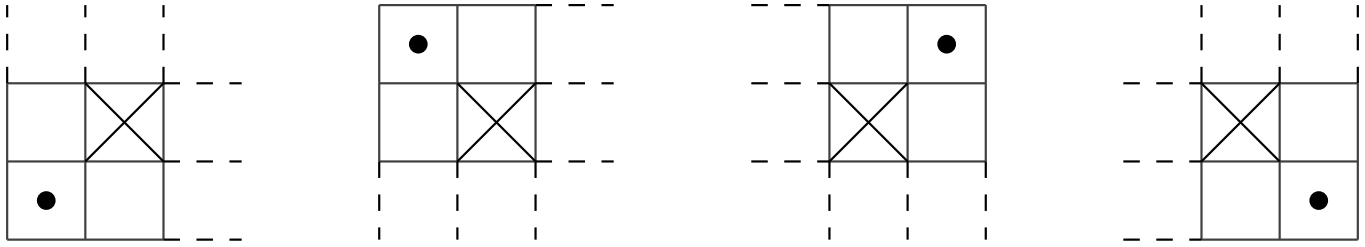

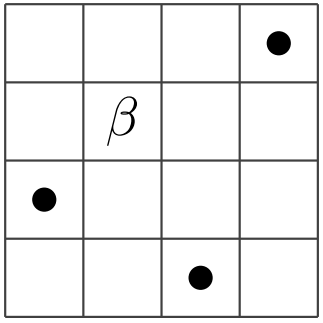

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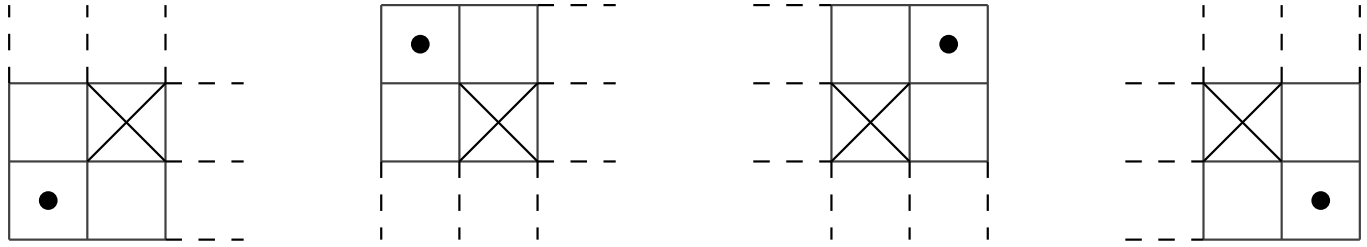

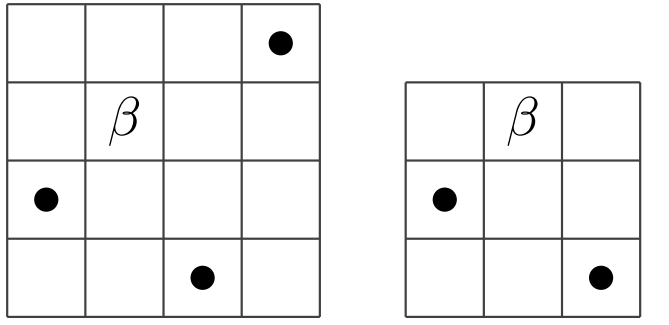

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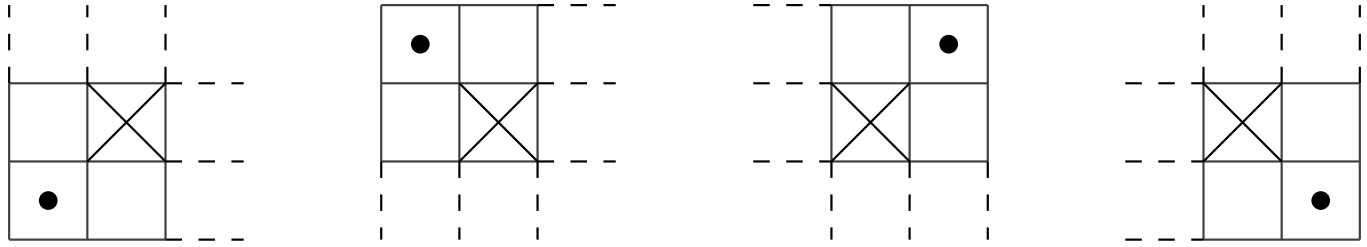

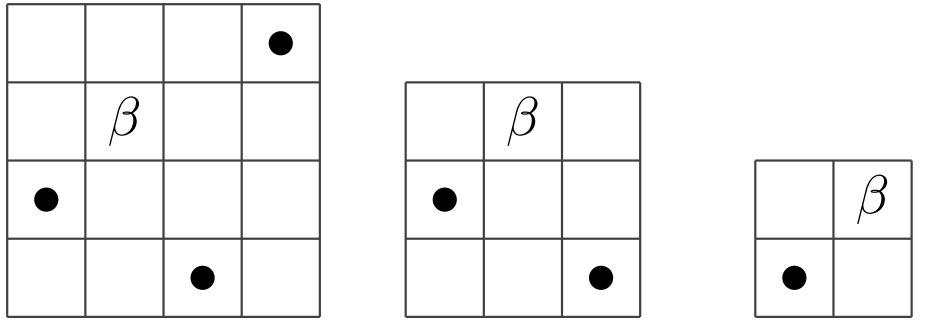

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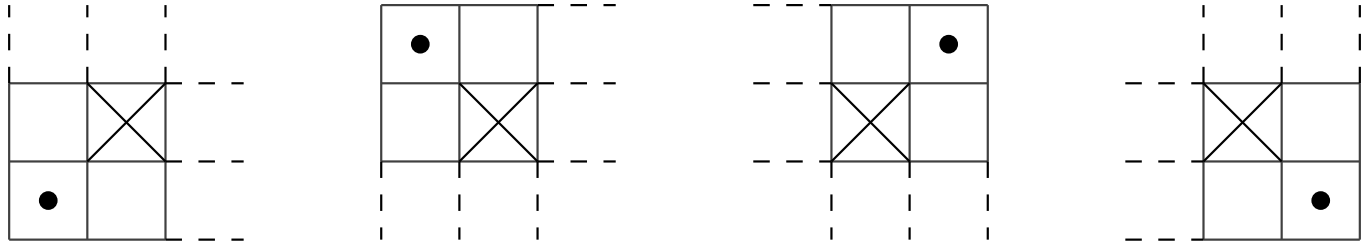

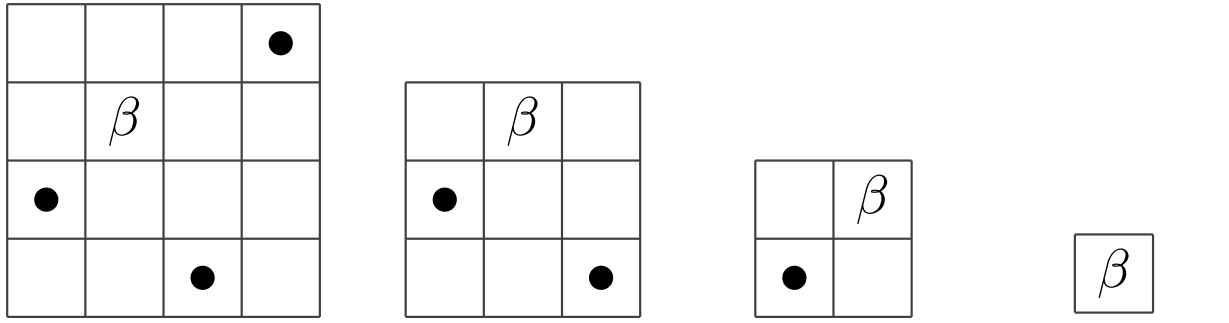

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Growth is at least exponential

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$$\text{Define } \pi^{(n)} = \begin{cases} 1 & \text{If } n = 1 \\ 12 & \text{If } n = 2 \\ 132 & \text{If } n = 3 \\ 2413 & \text{If } n = 4 \\ 2413 \circledcirc \pi^{(n-4)} & \text{Otherwise} \end{cases}$$

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The Open University

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Theorem

For all n, $\max_\mu(n) \geq 2^{\lfloor n/4 \rfloor - 1}$.

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Experimental results up to length 12 indicates that for **most** permutations,

$$\mu[\alpha \circledcirc_{i,j} \beta] \in \{\pm 2\mu[\beta], \pm\mu[\beta], 0\}$$

Future directions...

Can we find a better lower bound for $\max_\mu(n)$?

What about an upper bound?
We know that $\max_\mu(n) \leq n!$. (Zeigler, 1991)

Conjecture

$$\max_\mu(n) \leq 2^n$$

Make the balloon operation more general. Then what can we say about generalised balloons: $\alpha \circledcirc_{i,j} \beta$?

Experimental results up to length 12 indicates that for **most** permutations,

$$\mu[\alpha \circledcirc_{i,j} \beta] \in \{\pm 2\mu[\beta], \pm\mu[\beta], 0\}$$

Further, it appears that $\mu[\alpha \circledcirc_{i,j} \beta] = \pm 2\mu[\beta]$ only if $\beta = \alpha \circledcirc_{i,j} \gamma$.

Finally…

Thank you!

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