

Endless Pattern-Avoiding Permutations

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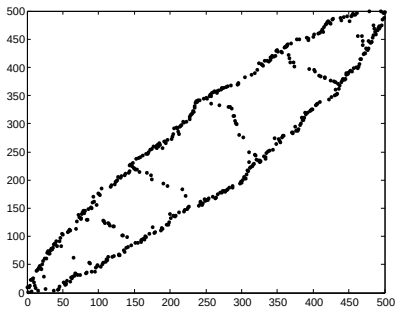
Permutation Patterns 2019

Joint work with Justin Troyka

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Outline

- ▶ Definitions: “Endless” $\leftrightarrow$ “bounded affine”
- ▶ Motivation (via statistical mechanics)
- ▶ Counting endless permutations
- ▶ Growth constants for endless pattern avoidance
- ▶ Endless 321-avoiding permutations: asymptotics (how a probabilistic view can help)
- ▶ A permuton-like result



A random 4231-avoiding permutation of length 500

A classical model of statistical mechanics is the **Ising model** of magnetism. Each atom in a lattice is in one of two states: $+$ or $-$. There is a joint probability distribution on all configurations in which spins of neighbouring atoms tend to agree.

+	+	+	-	-	+	-
-	+	+	-	-	-	+
+	+	+	-	+	+	-
+	-	-	+	-	-	+
+	-	-	+	+	-	-
-	+	+	-	+	-	+
+	+	+	+	-	+	+

Here is an example for $\mathbb{Z}^2$.

We first define the Ising model distribution in an $N \times N$ square, with 2^{N^2} configurations.

The object of interest is the $N \rightarrow \infty$ limit, a distribution on infinite configurations.

The model on finite squares can be studied numerically, but *boundary effects* can bias conclusions about the infinite case. One common strategy is to study squares with periodic boundary conditions.

Definition: a) An *affine permutation of period N* is a bijection $\omega : \mathbb{Z} \rightarrow \mathbb{Z}$ such that

$$\omega(i + N) = \omega(i) + N \quad \text{for every } i \in \mathbb{Z}, \text{ and}$$

$$\sum_{i=1}^N \omega(i) = \sum_{i=1}^N i \quad (\text{a “centering” condition}).$$

(b) An affine permutation ω of period N is said to be *bounded* if

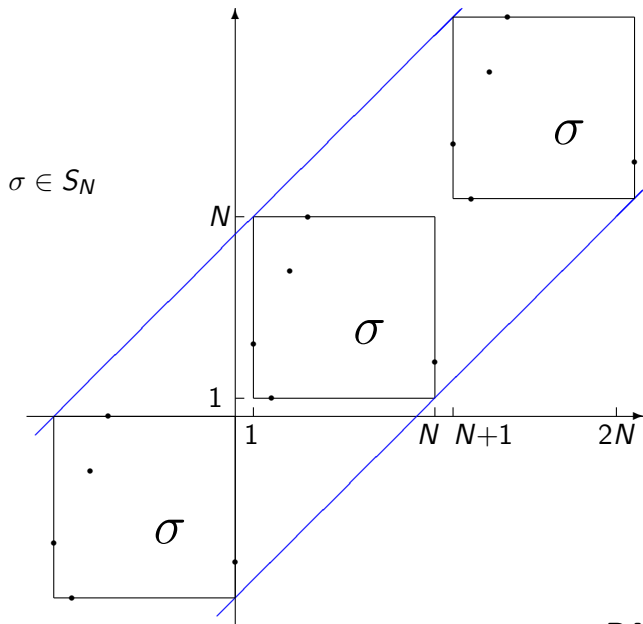
$$|\omega(i) - i| < N \quad \text{for every } i \in \mathbb{Z}.$$

We write $\mathbf{BA}_N$ for the set of bounded affine permutations of period N .

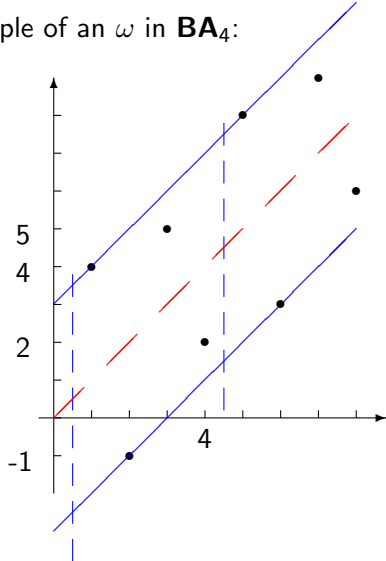
Observe that for any (ordinary) permutation $\sigma \in S_N$, the periodic extension of σ to a doubly infinite direct sum

$$\cdots \oplus \sigma \oplus \sigma \oplus \sigma \oplus \cdots$$

is in $\mathbf{BA}_N$.



A different example of an ω in $\mathbf{BA}_4$:



$$(\omega(1), \omega(2), \omega(3), \omega(4)) = (4, -1, 5, 2)$$

$$\omega(6) = 3 \quad \omega(-1) = 1 \quad \omega(-3) = 0$$

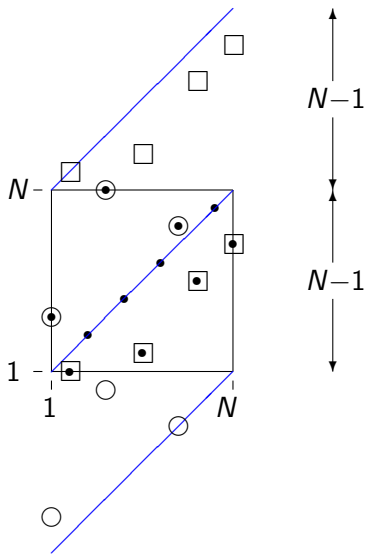
Also note that if ω is an affine permutation of period N , then it “projects congruentially” to a permutation σ in S_N via

$$\sigma(i) \equiv \omega(i) \pmod{N} \quad (i = 1, \dots, N).$$

Theorem

$$|\mathbf{BA}_N| \sim N! \times 2^N N^{-1/2} \sqrt{\frac{3}{2\pi e}} \quad \text{as } N \rightarrow \infty.$$

The (not quite right) idea of the proof:



Let $s_{N,j}$ be the number of permutations in S_N with exactly j fixed points. Then

$$|\mathbf{BA}_N| \stackrel{?}{=} \sum_{j=0}^N s_{N,j} 2^{N-j}$$

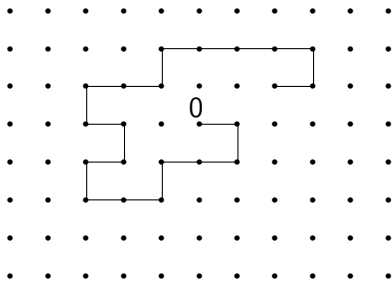
No... we need to enforce the centering condition.

Digression: Self-Avoiding Walks

A *self-avoiding walk* (SAW) in a given lattice is a path that revisits no vertex.

The picture shows a 21-step SAW in the square lattice.

Model of polymer configurations in statistical physics.



For a given lattice, let $\mathbf{SAW}_N$ be the set of N -step SAWs that start at the origin. Thus, for the square lattice,

$$|\mathbf{SAW}_1| = 4, \quad |\mathbf{SAW}_2| = 12, \quad \dots \quad |\mathbf{SAW}_{100}| = ?? \approx 10^{42}$$

Some Comparisons

Permutations

Beautiful theory

$\cap$ pattern-avoidance $\implies$

Can't do much!

Random walks

Beautiful theory

$\cap$ self-avoidance $\implies$

Can't do much!

Pattern-avoiding permutations have the *Stanley-Wilf limit*:

$$|\mathbf{Av}_N(\tau)| |\mathbf{Av}_M(\tau)| \leq |\mathbf{Av}_{N+M}(\tau)| \implies \exists \lim_{n \rightarrow \infty} |\mathbf{Av}_n(\tau)|^{1/n} =: L(\tau)$$

Self-avoiding walks have the *connective constant*:

$$|\mathbf{SAW}_N| |\mathbf{SAW}_M| \geq |\mathbf{SAW}_{N+M}| \implies \exists \lim_{n \rightarrow \infty} |\mathbf{SAW}_n|^{1/n} =: \mu$$

(both are problem-dependent)

Important unproven fact: for any “nice” lattice,

$$|\mathbf{SAW}_N| \sim A N^g \mu^N \quad (N \rightarrow \infty).$$

Here μ and A depend on the lattice, but g *depends only on the dimension of the lattice*.

E.g. in two dimensions,

$$\mu_{\text{Square}} \approx 2.638$$

$$\mu_{\text{Hex}} = \sqrt{2 + \sqrt{2}} \quad (\text{Nienhuis 1982; Duminil-Copin \& Smirnov 2012})$$

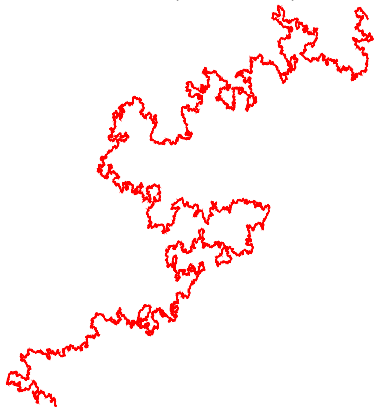
$$g = \frac{11}{32} \quad (\text{Nienhuis})$$

Contrast the situation with pattern-avoiding permutations:

$$|\mathbf{Av}_N(\mathbf{1234})| \asymp N^{-4} 9^N, \quad |\mathbf{Av}_N(\mathbf{1342})| \asymp N^{-5/2} 8^N,$$

$$|\mathbf{Av}_N(\mathbf{4231})| \stackrel{??}{\asymp} e^{-c\sqrt{N}} L^N \quad (\text{numerically, esp Guttman et al})$$

SAW in plane - 1,000,000 steps



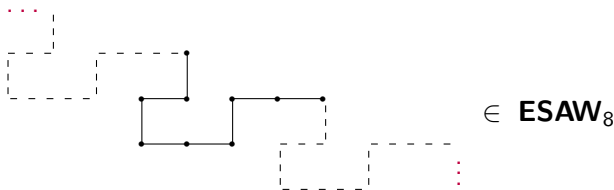
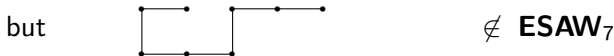
Generated randomly
by Tom Kennedy
(2002)

The ends of the walk are “special.” In the plane, we expect

Number of N -step self-avoiding cycles $\asymp N^{-5/2} \mu^N$

Number of N -step self-avoiding “bridges” $\asymp N^{-7/16} \mu^N$

To avoid all endpoint issues and focus on the bulk of the walk, Nathan Clisby (2013) proposed the class of “endless self-avoiding walks” (ESAWs). An N -step ESAW is an N -step SAW whose doubly infinite periodic extension is still self-avoiding; for example



Clisby’s conjecture: $|\mathbf{ESAW}_N| \sim A_e \mu^N$ (partly proven)

$\Rightarrow$ useful for estimating μ

Nathan asked me if this idea could be used for **Av(4231)**.

Endless Pattern-Avoiding Permutations

For a fixed pattern $\tau \in S_k$, let $\mathbf{AvBA}_N(\tau)$ be the set of all bounded affine permutations that avoid τ .

E.g. if $\tau_1 > \tau_k$ (or more generally if τ is sum indecomposable) and $\sigma \in \mathbf{Av}_N(\tau)$, then $\mathbf{AvBA}_N(\tau)$ contains $\cdots \oplus \sigma \oplus \sigma \oplus \sigma \oplus \cdots$, as well as all of its shifts. It follows that

$$\overline{\text{gr}}(\mathbf{AvBA}(\tau)) := \limsup_{n \rightarrow \infty} |\mathbf{AvBA}_n(\tau)|^{1/n} \geq L(\tau).$$

Henceforth we'll assume that τ is sum indecomposable.

Is the above inequality actually an equality?

In fact, if τ is 312 or 231 or 3142 or 2413 or $\dots$, then these are the only members of $\mathbf{AvBA}_N(\tau)$. Moreover, all affine permutations avoiding 312 (say) are automatically bounded. For details, see Justin Troyka's talk.

Therefore, for $\tau = 312, 231, 3142, 2413, \dots$, we see that

$$\text{gr}(\mathbf{AvBA}(\tau)) := \lim_{n \rightarrow \infty} |\mathbf{AvBA}_n(\tau)|^{1/n} = L(\tau).$$

Some more things we know:

$$(1) \quad \overline{\text{gr}}(\mathbf{AvBA}(\tau)) \leq 2L(\tau) \quad \text{for every } \tau;$$

but we can't prove that the limit $\text{gr}(\mathbf{AvBA}(\tau))$ always exists.

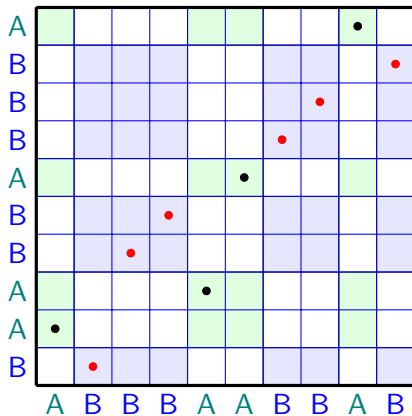
$$(2) \quad \text{gr}(\mathbf{AvBA}(\tau)) = L(\tau) \quad \text{for } \tau = \mathbf{k(k-1)(k-2) \dots 21}.$$

We shall describe the case $k = 3$ in more detail:

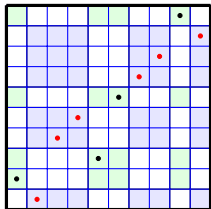
Theorem

$$|\mathbf{AvBA}_N(\mathbf{321})| \sim \frac{4^N \sqrt{N}}{2\sqrt{\pi}} \sim \frac{N^2}{2} |\mathbf{Av}_N(\mathbf{321})|.$$

Basic property: Every member of $\mathbf{AvBA}_N(\mathbf{321})$ can be partitioned into two increasing (periodic) sequences.



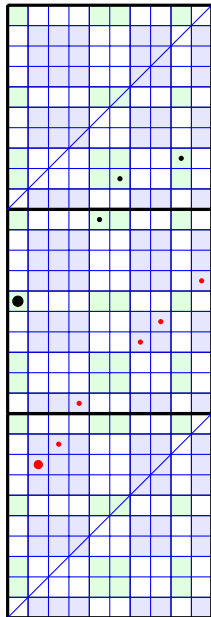
pick $\Delta \in \mathbb{Z}$; e.g. $\Delta = +2$



Shift black vertically Δ green squares

Shift red vertically $-\Delta$ blue squares

May exceed bounds, but still centered!



Upper bound:

$$\begin{aligned} |\mathbf{AvBA}_N(\mathbf{321})| &\leq \frac{1}{2} \sum_{k=1}^N \binom{N}{k}^2 (2k) \\ &= \frac{N}{2} \binom{2N}{N} = \frac{N(N+1)}{2} |\mathbf{Av}_N(\mathbf{321})|. \end{aligned}$$

We claim that this bound is asymptotically tight.

For the corresponding lower bound, we must show that

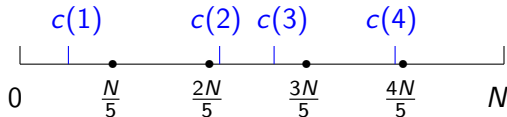
- most shifted configurations do not cross the blue lines, and
- multiple counting is negligible.

Key idea: Approximate uniformity of random subsets
("Don't worry, be random")

Approximate Uniformity

Notation:

For a finite set C , let $c(\ell)$ be the ℓ^{th} smallest element of C ($\ell = 1, \dots, |C|$)



Let $R > 0$. Define the collection $\mathbf{Unif}^{*R}(k, N)$ of k -element subsets of $[1, N]$ that are “within distance R of being uniformly distributed”:

$$\mathbf{Unif}^{*R}(k, N) = \left\{ C \subseteq [1, N] : |C| = k, \right. \\ \left. \left| c(\ell) - \frac{\ell N}{k+1} \right| \leq R \quad \forall \ell = 1, \dots, k \right\}$$

(Repeating the definition:)

$$\mathbf{Unif}^{*R}(k, N) = \left\{ C \subseteq [1, N] : |C| = k, \right. \\ \left. \left| c(\ell) - \frac{\ell N}{k+1} \right| \leq R \quad \forall \ell = 1, \dots, k \right\}$$

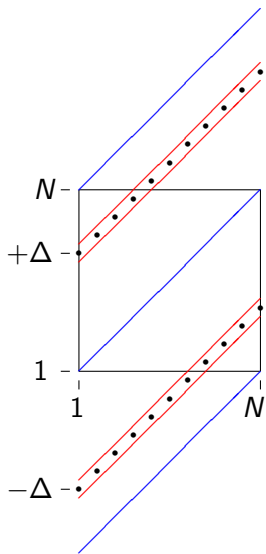
If the green sets along the horizontal and vertical sides of $[1, N]^2$ are both in $\mathbf{Unif}^{*R}(k, N)$, then the shifted black points are all within $4R$ of the line

$$y = x + \left(\frac{\Delta}{k+1} \right) N.$$

Lemma: Let $\theta \in (0, \frac{1}{2})$. Then for $N > N_0(\theta)$ and $\theta < \frac{k}{N} < 1 - \theta$,

$$\frac{\mathbf{Unif}^{*R}(k, N)}{\binom{N}{k}} \geq 1 - \frac{\text{Const.}(\theta) N^{3/2}}{R^2}.$$

(Useful for $R = \epsilon N$.) Proof uses Kolmogorov's Inequality.

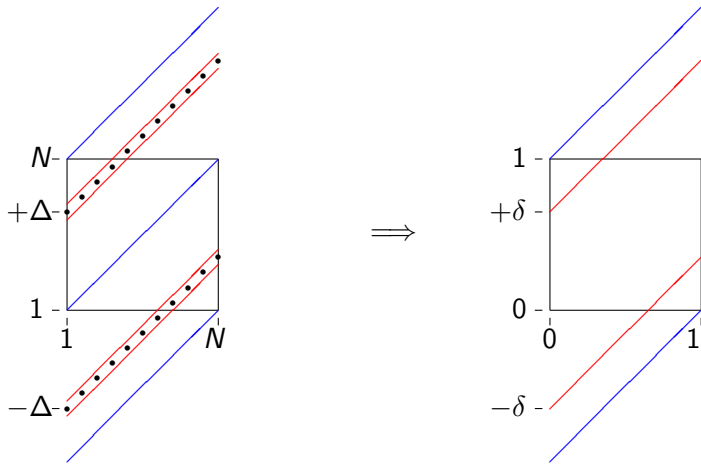


Lower bound:

$$\begin{aligned}
 & |\mathbf{A} \mathbf{v} \mathbf{B} \mathbf{A}_N(\mathbf{321})| \\
 & \geq \frac{1}{2} \sum_k^* \left[\binom{N}{k} \left(1 - O\left(\frac{1}{\sqrt{N}}\right) \right) \right]^2 (2-4\epsilon)k \\
 & \quad \left(\text{"}\sum_k^*\text{" : restrict to } \frac{1}{2} - \epsilon < \frac{k}{N} < \frac{1}{2} + \epsilon \right) \\
 & = \frac{1}{2} \sum_{k=1}^N \binom{N}{k}^2 (2k) \left(1 - o(\epsilon) - O\left(\frac{1}{\sqrt{N}}\right) \right)
 \end{aligned}$$

... which asymptotically matches the upper bound.

But wait, there's more! Scale our plot by $1/N$:



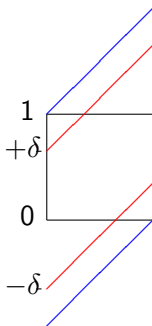
Here δ is a uniformly distributed random variable on $[0, 1]$.

Theorem: For $z \in [-1, 1]$, let λ_z be the probability measure in the plane that is uniformly distributed on the line segment joining $(0, z)$ to $(1, z + 1)$.

Let δ be a random variable uniformly distributed on $[0, 1]$.

Let μ_N be the scaled down empirical measure of a member of **AvBA_N(321)** chosen uniformly at random.

Then μ_N converges weakly to the random measure $\frac{1}{2}(\lambda_\delta + \lambda_{-\delta})$ as $N \rightarrow \infty$.



The measure $\frac{1}{2}(\lambda_\delta + \lambda_{-\delta})$

Idea of proof: Bound the Wasserstein distance.

For longer decreasing patterns, we think we can prove the following:

Fix $m \geq 3$. Then, as $N \rightarrow \infty$,

$$(1) \quad |\mathbf{AvBA}_N(\mathbf{m(m-1) \cdots 21})| \asymp N^{\frac{m-2}{2}} (m-1)^{2N}$$

$$\left(\text{c.f. } |\mathbf{Av}_N(\mathbf{m(m-1) \cdots 21})| \asymp N^{m-\frac{m^2}{2}} (m-1)^{2N} \right)$$

(2) Let μ_N be the scaled down empirical measure of a member of $\mathbf{AvBA}_N(\mathbf{m \cdots 21})$ chosen uniformly at random.

Then μ_N converges weakly to the random measure

$$\frac{1}{m-1} (\lambda_{\delta(1)} + \cdots + \lambda_{\delta(m-1)}),$$

where $(\delta(1), \dots, \delta(m-1))$ is chosen uniformly at random from $\{\vec{x} \in [-1, 1]^{m-1} : \sum_i x_i = 0\}$.

A Few Open Questions

- (i) Does $\overline{\text{gr}}(\mathbf{AvBA}(\tau))$ equal the Stanley-Wilf value $L(\tau)$ for every (sum indecomposable) τ ?
- (ii) If not, does $\text{gr}(\mathbf{AvBA}(\tau))$ exist?
- (iii) Can we learn anything about $\mathbf{Av}(\tau)$ from $\mathbf{AvBA}(\tau)$... or is this just an interesting side trip?

Why (i) may be hard:

For a given $\sigma \in \mathbf{Av}_N(\tau)$, how many $\omega \in \mathbf{AvBA}_N(\tau)$ can have the same reduced form σ for $(\omega(1), \dots, \omega(N))$?

Is it $\exp[o(N)]$ for most σ 's?

But for $\tau = \mathbf{321}$, most ω 's have atypical reduced form!

(E.g. ascent heights of order N)

