On pattern-avoiding Fishburn permutations[1]

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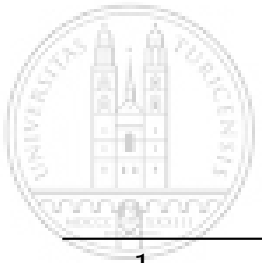
Permutation Patterns 2019



Universität Zürich[UZH]

Joint work with Mike Weiner

[1] Preprint arXiv:1812.01682, December 2018

Motivation: Fishburn numbers $\xi(n)$, defined by

$$\sum_{n=0}^{\infty} \xi(n) q^n = 1 + \sum_{n=1}^{\infty} \prod_{j=1}^{n} (1 - (1-q)^j).$$

Sequence A022493 in the OEIS:

$$1, 2, 5, 15, 53, 217, 1014, 5335, 31240, 201608, 1422074, \ldots,$$

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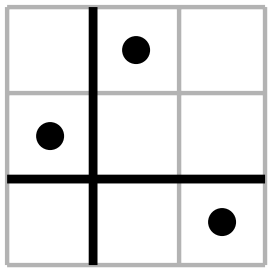
Bivincular pattern $(231, \{1\}, \{1\})$

A permutation $\pi \in S_n$ is said to contain the bivincular pattern $(231, \{1\}, \{1\})$ if there are positions $i < k$ with $\pi(i) > 1$, $\pi(k) = \pi(i) - 1$, such that the subsequence $\pi(i)\pi(i+1)\pi(k)$ forms a 231 pattern:



Bold lines indicate adjacent entries
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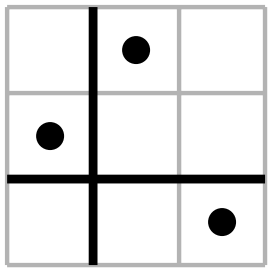

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For $n = 4$, there are 9 permutations containing such a pattern:

1342, 2314, 2341, 2413, 2431, 3241, 3412, 3421, 4231.

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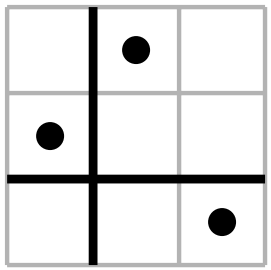

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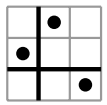
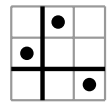
The permutation 3142 contains 231 but avoids .

Fishburn permutations

Let $\mathscr{F}_n$ denote the class of permutations in S_n that avoid .
M. Bousquet-Mélou et. al. (2010) proved that $\mathscr{F}_n$ is enumerated by $\xi(n)$, so we call the elements of $\mathscr{F}_n$ *Fishburn permutations*.

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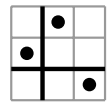
Let $\mathscr{F}_n$ denote the class of permutations in S_n that avoid .
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$$\mathrm{Av}_n(231) \subset \mathscr{F}_n$$

For example,

$$\mathscr{F}_4 = \{1234, 1243, 1324, 1423, 1432, 2134, 2143, 3124,$$
$$3142, 3214, 4123, 4132, 4213, 4312, 4321\}$$

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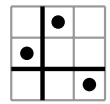
Let $\mathscr{F}_n$ denote the class of permutations in S_n that avoid .
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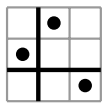
$$\mathscr{F}_4 = \{1234, 1243, 1324, 1423, 1432, 2134, 2143, 3124, 3142, 3214, 4123, 4132, 4213, 4312, 4321\}$$

It is easier to avoid  than it is to avoid 231.

Pattern avoidance

Let $\mathscr{F}_n(\sigma)$ denote the class of Fishburn permutations that avoid the pattern σ. Our goal is to study $F_n(\sigma) = |\mathscr{F}_n(\sigma)|$ for classical patterns of size 3 or 4.

Since $\mathrm{Av}_n(231) \subset \mathscr{F}_n$ and $\mathscr{F}_n(231) \subset \mathrm{Av}_n(231)$, we get

$$\mathscr{F}_n(231) = \mathrm{Av}_n(231), \text{ thus } F_n(231) = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

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Theorem

For $\sigma \in \{123, 132, 213, 312\}$, we have $F_n(\sigma) = 2^{n-1}$.

Theorem

$\mathscr{F}_n(321)$ is in bijection with the set of n-Dyck paths that avoid the subpath UUDU.

[3] Ordered trees and the inorder traversal, *Discrete Math.* **306** (2006)

Theorem

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By a result from Sapounakis/Tasoulas/Tsikouras,[3] we then have

$$F_n(321) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \frac{(-1)^j}{n-j} \binom{n-j}{j} \binom{2n-3j}{n-j+1}.$$

This is sequence A105633 (1, 2, 4, 9, 22, 57, 154, 429, 1223,...).

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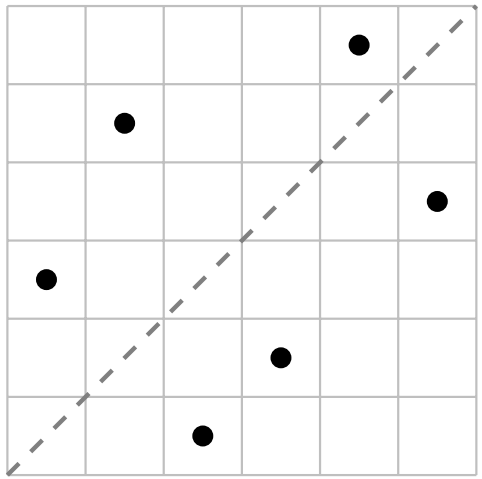
Av(321) to Dyck paths

For example, for $\pi = 351264 \in \mathrm{Av}_6(321)$ we get:



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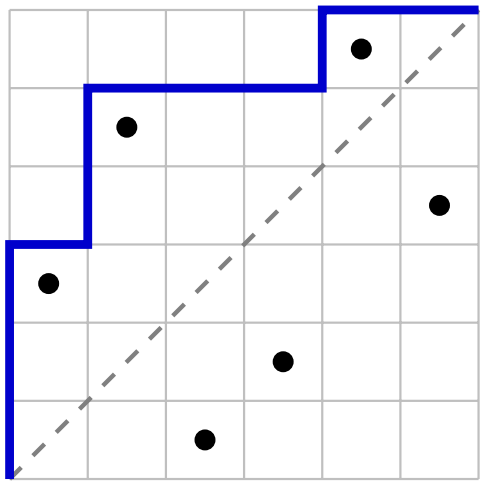

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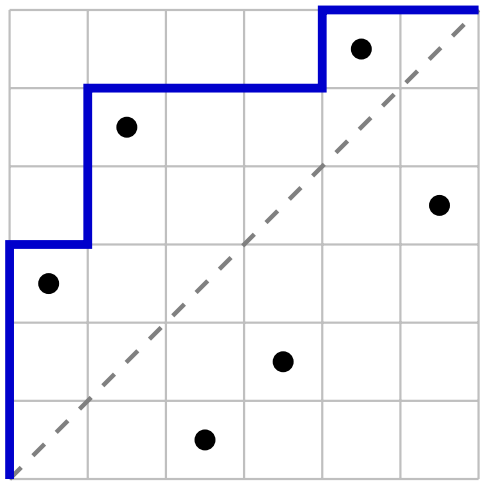

Not Fishburn!

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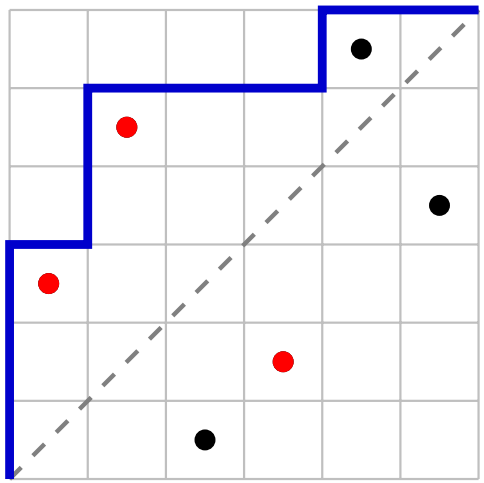

Note that an ascent $\pi_i < \pi_{i+1}$ in $\pi \in \mathrm{Av}_n(321)$ with $k = \pi_{i+1} - \pi_i$ generates UDU^k in the associated Dyck path P, and if $\pi_i - 1 = \pi_j$ for some $j > i + 1$, then P must necessarily contain $UUDU^k$.

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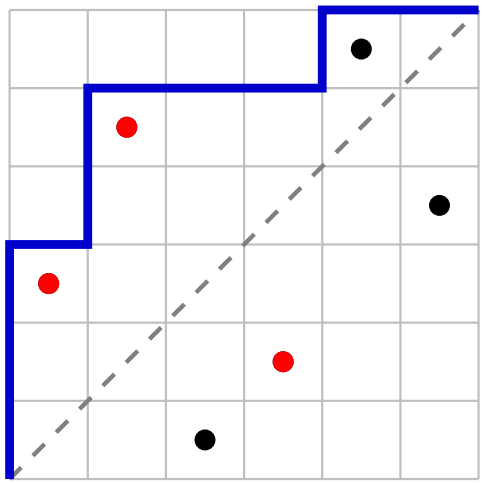

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Avoiding classical patterns of size 3

Pattern σ	$\lvert\mathscr{F}_n(\sigma)\rvert$	OEIS
123, 132, 213, 312	1, 2, 4, 8, 16, 32, 64, 128, 256, 512, ...	A000079
231	1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...	A000108
321	1, 2, 4, 9, 22, 57, 154, 429, 1223, 3550, ...	A105633

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123	1, 1, 2, 5, 12, 27, 58, 121, 248, 503, ...	A000325
132, 213	1, 1, 2, 4, 8, 16, 32, 64, 128, 256, ...	A000079
231	1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...	A000108
312	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...	A000012
321	1, 1, 1, 2, 5, 13, 35, 97, 275, 794, ...	A082582

Avoiding patterns of size 4

Pattern σ	$\lvert\mathscr{F}_n(\sigma)\rvert$	OEIS
1342	1, 2, 5, 15, 51, 188, 731, 2950,...	A007317
1432	1, 2, 5, 14, 43, 142, 495, 1796,...	
2314	1, 2, 5, 15, 52, 200, 827, 3601,...	
2341	1, 2, 5, 15, 52, 202, 858, 3910,...	
3412	1, 2, 5, 15, 52, 201, 843, 3764,...	A202062*
3421	1, 2, 5, 15, 52, 203, 874, 4076,...	
4123	1, 2, 5, 14, 42, 133, 442, 1535,...	
4231	1, 2, 5, 15, 52, 201, 843, 3765,...	
4312	1, 2, 5, 14, 43, 143, 508, 1905,...	
4321	1, 2, 5, 14, 45, 162, 639, 2713,...	
2413, 2431, 3241	1, 2, 5, 15, 52, 201, 842, 3745,...	
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*w/L. Pudwell (in progress)

Catalan class

Theorem

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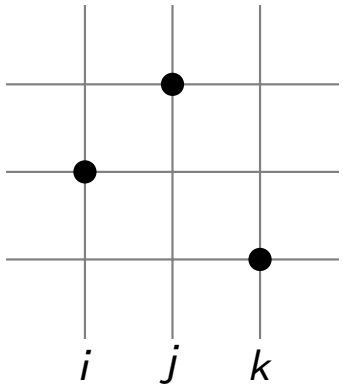
$$\begin{array}{ccccccc}
\mathscr{F}_n(3124) & \xleftarrow{\alpha_1} & \mathscr{F}_n(3142) & \xrightarrow{\gamma} & \mathscr{F}_n(2143) & \xleftarrow{\phi} & \mathscr{F}_n(2134) \\
\downarrow{\scriptstyle \alpha_2} & & & & & & \\
\mathscr{F}_n(1324) & \xrightarrow{\alpha} & \mathscr{F}_n(1234) & \xrightarrow{\phi} & \mathscr{F}_n(1243) & \xleftarrow{\alpha} & \mathscr{F}_n(1423)
\end{array}$$

Theorem

We have $\mathscr{F}_n(3142) = \mathscr{F}_n(231)$, *hence* $|\mathscr{F}_n(3142)| = C_n$.

Proof. Clearly, $\mathscr{F}_n(231) \subseteq \mathscr{F}_n(3142)$.
Suppose $\pi \in \mathscr{F}_n(3142)$ contains the pattern 231 and let $i < j < k$ be the positions of the most-left closest 231 pattern contained in π:



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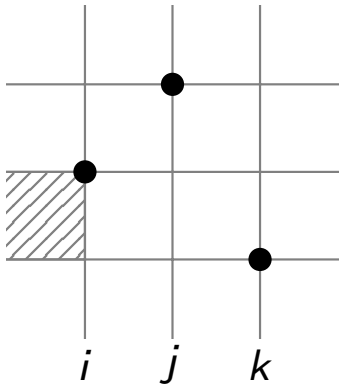

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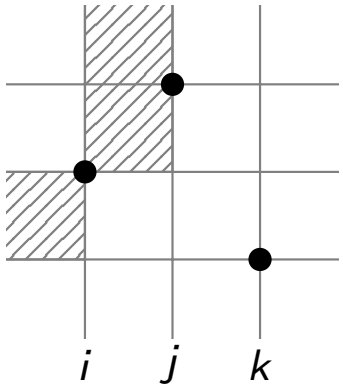

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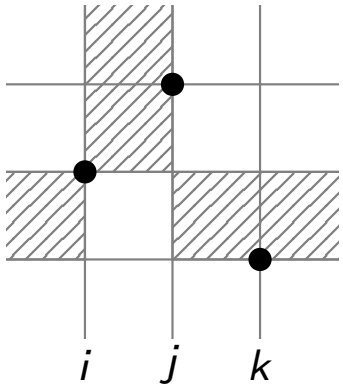

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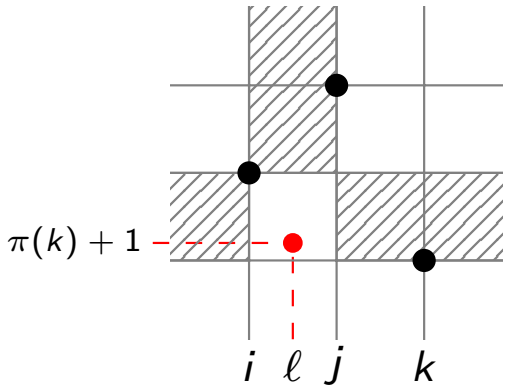

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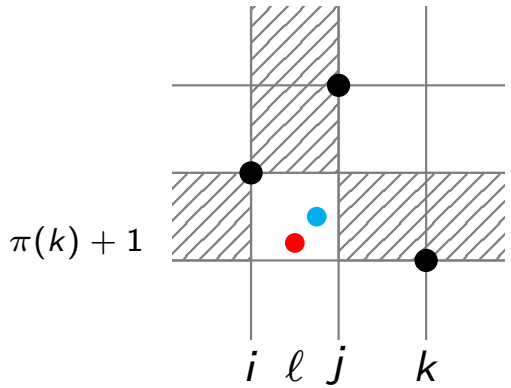

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We have $\mathscr{F}_n(3142) = \mathscr{F}_n(231)$, *hence* $|\mathscr{F}_n(3142)| = C_n$.

Proof. Clearly, $\mathscr{F}_n(231) \subseteq \mathscr{F}_n(3142)$.
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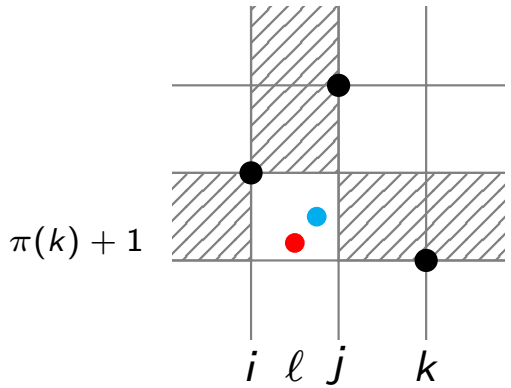

Fishburn violation

Theorem

We have $\mathscr{F}_n(3142) = \mathscr{F}_n(231)$, *hence* $|\mathscr{F}_n(3142)| = C_n$.

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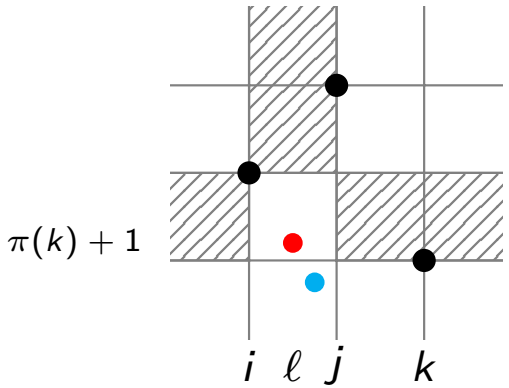

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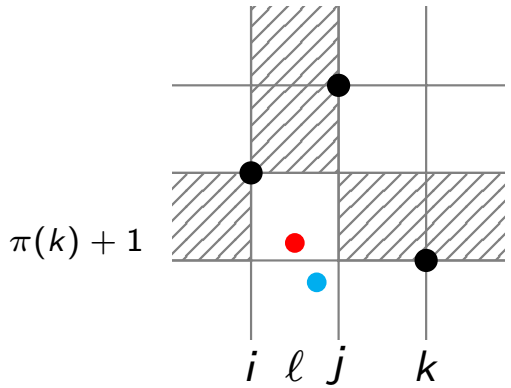
We have $\mathscr{F}_n(3142) = \mathscr{F}_n(231)$, *hence* $|\mathscr{F}_n(3142)| = C_n$.

Proof. Clearly, $\mathscr{F}_n(231) \subseteq \mathscr{F}_n(3142)$.
Suppose $\pi \in \mathscr{F}_n(3142)$ contains the pattern 231 and let $i < j < k$ be the positions of the most-left closest 231 pattern contained in π:

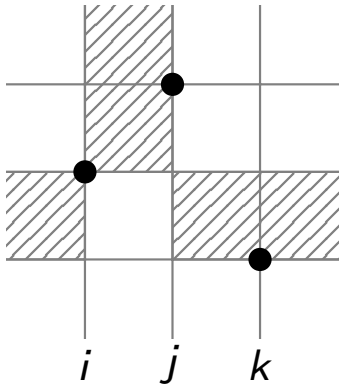


3142 violation

Theorem

We have $\mathscr{F}_n(3142) = \mathscr{F}_n(231)$, *hence* $|\mathscr{F}_n(3142)| = C_n$.

Proof. Clearly, $\mathscr{F}_n(231) \subseteq \mathscr{F}_n(3142)$.
Suppose $\pi \in \mathscr{F}_n(3142)$ contains the pattern 231 and let $i < j < k$ be the positions of the most-left closest 231 pattern contained in π:



In conclusion, no permutation $\pi \in \mathscr{F}_n(3142)$ contains a 231 pattern. Thus $\mathscr{F}_n(3142) \subseteq \mathscr{F}_n(231)$. □

Future directions...

- Pattern avoiding ascent sequences

Future directions...

- Pattern avoiding ascent sequences
- Indecomposable permutations of size 4

Indecomposable permutations

Pattern σ	$\lvert\mathscr{F}_n^{\mathsf{ind}}(\sigma)\rvert$	OEIS
1234	1, 1, 2, 6, 22, 85, 324, 1204,...	
1243, 2134	1, 1, 2, 6, 21, 75, 266, 938,...	A289597(?)
1324	1, 1, 2, 6, 22, 84, 317, 1174, ...	
1342	1, 1, 2, 6, 22, 88, 367, 1568, ...	A165538
1423, 3124	1, 1, 2, 6, 20, 68, 233, 805, ...	A279557
1432	1, 1, 2, 6, 20, 71, 263, 1002, ...	
2143	1, 1, 2, 6, 19, 62, 207, 704, ...	A026012
2314	1, 1, 2, 6, 23, 99, 450, 2109, ...	
2341	1, 1, 2, 6, 22, 91, 409, 1955, ...	
2413, 2431, 3241	1, 1, 2, 6, 22, 90, 395, 1823, ...	A165546(?)
3142	1, 1, 2, 5, 14, 42, 132, 429, ...	A000108

Indecomposable permutations

Pattern σ	$\lvert\mathscr{F}_n^{\mathsf{ind}}(\sigma)\rvert$	OEIS
3214	1, 1, 2, 6, 20, 72, 275, 1096, ...	
3412	1, 1, 2, 6, 22, 90, 396, 1840, ...	
3421	1, 1, 2, 6, 22, 92, 423, 2088, ...	
4123	1, 1, 2, 5, 14, 43, 143, 507, ...	
4132, 4213	1, 1, 2, 5, 15, 51, 188, 732, ...	
4231	1, 1, 2, 6, 22, 90, 396, 1841, ...	
4312	1, 1, 2, 5, 15, 51, 188, 733, ...	
4321	1, 1, 2, 5, 17, 66, 279, 1256, ...	

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Thank you!