

Permutation patterns: gamma-positivity and (-1) -phenomenon

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If $A = \{a_k\}_{k=0}^n$ is a finite sequence of real numbers, then

- A is **unimodal** if there is an index $0 \leq j \leq n$ such that

$$a_0 \leq \cdots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \cdots \geq a_n.$$

- A is **log-concave** if

$$a_j^2 \geq a_{j-1}a_{j+1}, \quad \text{for all } 1 \leq j \leq n.$$

- the generating polynomial, $p_A(x) := a_0 + a_1x + \cdots + a_nx^n$, is called **real rooted** if all its zeros are real.

Example

The n -th row of Pascal's triangle $\{\binom{n}{k}\}_{k=0}^n: \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}$:

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & 1 & & 1 & \\ & & 1 & & 2 & & 1 \\ 1 & & 3 & & 3 & & 1 \\ & \dots & \dots & & & & \end{array}$$

The explicit formula $\binom{n}{k} = n!/k!(n-k)!$ implies

$$\frac{\binom{n}{k}^2}{\binom{n}{k-1}\binom{n}{k+1}} = \frac{(k+1)(n-k+1)}{k(n-k)} > 1.$$

A basic theorem

Let $A = \{a_k\}_{k=0}^n$ be a finite sequence of nonnegative numbers.

- (a) If $p_A(x)$ is real-rooted, then the sequence $A' := \{a_k / \binom{n}{k}\}_{k=0}^n$ is log-concave.
- (b) If A' is log-concave, then so is A .
- (c) If A is log-concave and positive, then A is unimodal.

Remark

We say A is *ultra-log-concave* if A' is log-concave.

real-roots $\implies$ *ultra-log-concavity* $\implies$ *log-concavity* $\implies$ *unimodality*

$$a_k^2 \geq a_{k-1} a_{k+1} \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right).$$

Unimodality and γ -positivity

The sequence $A = \{a_k\}_{k=0}^n$ is **symmetric** (or **plindromic**) with center of symmetry $n/2$ if $a_k = a_{n-k}$ for $0 \leq k \leq n/2$.

The linear space of polynomials $h(x) = \sum_{k=0}^n a_k x^k \in \mathfrak{R}[x]$ which are symmetric with center of symmetry $n/2$ has a basis

$$B_n := \{x^k(1+x)^{n-2k}\}_{k=0}^{\lfloor n/2 \rfloor}.$$

Indeed, we have the so-called Waring's formula:

$$x^n + y^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (xy)^j (x+y)^{n-2j}.$$

Proof of the symmetric expansion

Let $\gamma_{n,j} = (-1)^j \frac{n}{n-j} \binom{n-j}{j}$, then

$$x^n + y^n = \sum_{j=0}^{n/2} \gamma_{n,j} (xy)^j (x + y)^{n-2j}.$$

As $a_j = a_{n-j}$ (symmetry), we have

$$\begin{aligned} a_j x^j + a_{n-j} x^{n-j} &= a_j x^j (1 + x^{n-2j}) \\ &= a_j x^j \sum_k \gamma_{n-2j,k} x^k (x + 1)^{n-2j-2k} \\ &= a_j \sum_k \gamma_{n-2j,k} x^{k+j} (x + 1)^{n-2(j+k)} \end{aligned}$$

Real-roots and symmetry

If $h(x) = \sum_{k=0}^n \gamma_k x^k (1+x)^{n-2k}$ with $\gamma_k \geq 0$ we say that h is **γ -nonnegative**. Since the binomial coefficients are unimodal, having a nonnegative γ -vector implies **unimodality** of $\{a_k\}_{k=0}^n$. Let Γ_+^n be the convex cone of polynomials that have nonnegative coefficients when expanded in B_n . Clearly

$$\Gamma_+^n \cdot \Gamma_+^m \subset \Gamma_+^{m+n}.$$

If $h(x) = \sum_{k=0}^n a_k x^k \in \mathfrak{R}_+[x]$ is symmetric with center of symmetry $n/2$. If all its zeros are real, then we may pair the negative zeros into reciprocal pairs

$$h(x) = Ax^k \prod_{i=1}^{\ell} (x+\theta_i)(x+1/\theta_i) = Ax^k \prod_{i=1}^{\ell} ((1+x)^2 + (\theta_i + 1/\theta_i - 2)x),$$

where $A > 0$. Since x and $(1+x)^2 + (\theta_i + 1/\theta_i - 2)x$ are in Γ_+^1 , we see that $h(x)$ is γ -nonnegative.

Eulerian polynomials

The eulerian polynomials $A_n(t)$ can be defined by

$$\sum_{n=0}^{\infty} A_n(t) \frac{x^n}{n!} = \frac{1-t}{1-te^{(1-t)x}}.$$

The first values of $A_n(t) := a_{n,1}t + a_{n,2}t^2 + \cdots + a_{n,n}t^n$:

$$A_1(t) = t,$$

$$A_2(t) = t + t^2,$$

$$A_3(t) = t + 4t^2 + t^3,$$

$$A_4(t) = t + 11t^2 + 11t^3 + t^4,$$

$$A_5(t) = t + 26t^2 + 66t^3 + 26t^4 + t^5.$$

Combinatorial interpretations

Let $\mathfrak{S}_n$ is the set of permutations on $\{1, \dots, n\}$.

$$\text{des } \sigma = \#\{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\},$$

$$\text{exc } \sigma = \#\{i \in [n] \mid \sigma(i) > i\},$$

Proposition

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{1+\text{des } \sigma} = \sum_{\sigma \in \mathfrak{S}_n} t^{1+\text{exc } \sigma}$$

The equidistribution $\text{exc} \sim \text{des}$ can be explained by Foata's transformation

Unimodal and symmetric property

Proposition

The Eulerian polynomial $A_n(t)$ is *symmetric* and has only *real roots*.

Hence $A_n(t)$ is γ -positive:

$$t = t,$$

$$t + t^2 = t \cdot (1 + t),$$

$$t + 4t^2 + t^3 = t \cdot (1 + t)^2 + 2t^2,$$

$$t + 11t^2 + 11t^3 + t^4 = t(1 + t)^3 + 8t^2(1 + t).$$

This formula implies both the symmetry and the unimodality of the Eulerian numbers.

Foata-Schützenberger, Foata-Strehl in 1970's

Let $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$ with $\sigma(0) = \sigma(n+1) = 0$. An integer $i \in [1, n]$ is

- a **peak** in σ if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$;
- a **double descent** if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.

Proposition

Let $a_{n,k}$ be the number of permutations of $[n]$ with k peaks and without double descent. Then

$$A_n(t) = \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} a_{n,k} t^k (1+t)^{n+1-2k}. \quad (1)$$

Example: If $n = 4$, $k = 2$, then $a_{n,k} = 8$:

2134, 3124, 4123, 1423, 2413, 3412, 1324, 2314

Derangement analogue

For $\sigma \in \mathfrak{S}_n$, a value $x = \sigma(i)$ ($i \in [n]$) is called

- a **cyclic peak** if $i = \sigma^{-1}(x) < x$ and $x > \sigma(x)$;
- a **cyclic double descent** if $i = \sigma^{-1}(x) > x$ and $x > \sigma(x)$;

Let $\mathfrak{D}_n$ be the set of **derangements** in $\mathfrak{S}_n$. Consider the derangement analogue of the Eulerian polynomials:

$$D_n(t) = \sum_{\sigma \in \mathfrak{D}_n} t^{\text{exc } \sigma} = D_{n,1}t + D_{n,2}t^2 + \cdots + D_{n,n-1}t^{n-1}.$$

Proposition (Shin-Zeng 2012)

Let $b_{n,k}$ be the number of derangements of $[n]$ with k cyclic peaks and without cyclic double descents. Then

$$D_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{n,k} t^k (1+t)^{n-2k}. \quad (2)$$

(-1) -phenomenon for Eulerian polynomials

We can take $t = -1$ and recover the following combinatorial interpretation of a classical identity involving the odd Euler number E_{2n+1}

$$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{des } \pi} = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{exc } \pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-1}{2}} E_n & \text{if } n \text{ is odd.} \end{cases}$$

Let $\pi = \pi(1)\pi(2) \cdots \pi(n)$ be a permutation, assume $\pi(n+1) = n+1$. A *descent* (resp. an *ascent*) in π is a triple $(i, \pi(i), \pi(i+1))$ such that $i \in [n]$ and $\pi(i) > \pi(i+1)$ (resp. $\pi(i) < \pi(i+1)$). Here $\pi(i)$ is called the *descent top* (resp. the *ascent bottom*) and $\pi(i+1)$ is called the *descent bottom* (resp. the *ascent top*).

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For all $1 \leq i \leq n$, $\pi(i)$ is called a **left-to-right maximum** if $\pi(i) = \max \{\pi(1), \pi(2), \dots, \pi(i)\}$. An ascent bottom $\pi(i)$ ($i = 1, \dots, n$) is called a **foremaximum** of π if it is at the same time a left-to-right maximum. Denote the number of foremaxima of π by $\text{fmax } \pi$.

For example, the foremaximums of $4157368 \in S_8$ are 5 and 8, $\text{fmax}(4157368) = 2$.

(-1) -phenomenon for Derangement polynomials

A parallel result for the even Euler number E_{2n} was first given by Roselle

$$\sum_{\pi \in \mathfrak{D}_n^*} (-1)^{\text{des } \pi} = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\text{exc } \pi} = \begin{cases} (-1)^{\frac{n}{2}} E_n & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

A permutation π is called **coderangement** if $\text{fmax } \pi = 0$. Let $\mathfrak{D}_n^*$ be the set of *coderangements* of length n .

Continued fraction of Catalan number

It is well known that **Catalan numbers** $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the coefficient of the following Stieltjes continued fraction expansion

$$\sum_{n=0}^{\infty} C_n z^n = \frac{1}{1 - \frac{z}{1 - \frac{z}{\ddots}}}.$$

We define the (q, t) -Catalan numbers $C_n(t, q)$ as the Taylor coefficients in the following continued fraction expansion

$$\sum_{n=0}^{\infty} C_n(t, q) z^n = \frac{1}{1 - \frac{z}{1 - \frac{tz}{1 - \frac{\ddots}{1 - \frac{q^{k-1}z}{1 - \frac{tq^{k-1}z}{1 - \frac{\ddots}{\ddots}}}}}}}$$

Vincular patterns

The number of occurrences of **vincular patterns** 31-2, 2-31, 2-13 and 13-2 in $\pi \in \mathfrak{S}_n$ are defined by

$$(31-2) \pi = \#\{(i, j) : i + 1 < j \leq n \text{ and } \pi(i + 1) < \pi(j) < \pi(i)\},$$

$$(2-31) \pi = \#\{(i, j) : j < i < n \text{ and } \pi(i + 1) < \pi(j) < \pi(i)\},$$

$$(2-13) \pi = \#\{(i, j) : j < i < n \text{ and } \pi(i) < \pi(j) < \pi(i + 1)\},$$

$$(13-2) \pi = \#\{(i, j) : i + 1 < j \leq n \text{ and } \pi(i) < \pi(j) < \pi(i + 1)\}.$$

we use $\mathfrak{S}_n(31-2)$ to denote the set of permutations of length n that avoid the vincular pattern 31-2.

Our result

The (q, t) -Catalan numbers $C_n(t, q)$ have the following **eight** interpretations

$$C_n(t, q) = \sum_{\pi \in \mathfrak{S}_n(\tau)} t^{\text{des } \pi} q^{\text{stat } \pi},$$

with τ being a pattern of length 3, and stat being a permutation statistic. Eight choices for the pair (τ, stat) are listed in Table 1.

Table: Eight choices for (τ, stat)

#	τ	stat	#	τ	stat
1	231	13-2	5	231	31-2
2	312	2-13	6	312	2-31
3	213	31-2	7	213	13-2
4	132	2-31	8	132	2-13

Define

$$\tilde{\mathfrak{S}}_{n,k}(213) := \{\pi \in \mathfrak{S}_n(213) : \text{dd } \pi = 0, \text{des } \pi = k\},$$

$$\tilde{\mathfrak{S}}_{n,k}(312) := \{\pi \in \mathfrak{S}_n(312) : \text{dd } \pi = 0, \text{des } \pi = k\},$$

$$\tilde{\mathfrak{S}}_{n,k}(132) := \{\pi \in \mathfrak{S}_n(132) : \text{dd}^* \pi = 0, \text{des } \pi = k\},$$

$$\tilde{\mathfrak{S}}_{n,k}(231) := \{\pi \in \mathfrak{S}_n(231) : \text{dd}^* \pi = 0, \text{des } \pi = k\}.$$

Note that we use the $*$ version to indicate the **initial condition** $\pi(0) = \pi(n+1) = n+1$, while the non- $*$ version means $\pi(0) = \pi(n+1) = 0$.

Theorem

For $n \geq 1$, the following γ -expansions formula holds true

$$C_n(t, q) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}(q) t^k (1+t)^{n-1-2k}, \quad (3)$$

where

$$\gamma_{n,k}(q) = \sum_{\pi \in \tilde{\mathfrak{S}}_{n,k}(213)} q^{(31-2)\pi} = \sum_{\pi \in \tilde{\mathfrak{S}}_{n,k}(312)} q^{(2-13)\pi} \quad (4)$$

$$= \sum_{\pi \in \tilde{\mathfrak{S}}_{n,k}(132)} q^{(2-31)\pi} = \sum_{\pi \in \tilde{\mathfrak{S}}_{n,k}(231)} q^{(13-2)\pi}. \quad (5)$$

Our result

The statistic **inversion number** is defined by

$$\text{inv } \pi = \#\{(i, j) : 1 \leq i < j \leq n \text{ and } \pi(i) > \pi(j)\}$$

Recall that Carlitz's **q -Catalan numbers** $C_n(q)$ are defined by

$$C_n(q) := C_n(q, q^2). \quad (6)$$

Theorem

For any $n \geq 1$,

$$\sum_{\pi \in \mathfrak{S}_n(321)} (-1)^{\text{exc } \pi} q^{\text{inv } \pi - \text{exc } \pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-q)^{\frac{n-1}{2}} C_{\frac{n-1}{2}}(q^2) & \text{if } n \text{ is odd,} \end{cases}$$
$$\sum_{\pi \in \mathfrak{D}_n(321)} (-1)^{\text{exc } \pi} q^{\text{inv } \pi} = \begin{cases} (-q)^{\frac{n}{2}} C_{\frac{n}{2}}(q^2) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

$$\text{MAD } \pi := \text{des } \pi + (31-2) \pi + 2 \cdot (2-31) \pi,$$

$$\text{fix } \pi := \sum_{1 \leq i \leq n} \chi(\pi(i) = i),$$

$$\text{wex } \pi := \text{exc } \pi + \text{fix } \pi,$$

where $\chi(A) = 1$ if A is true and 0 otherwise.

$$\text{cros } \pi = \sum_{i=1}^n \#\{j \in [n] | j < i < \pi(j) < \pi(i) \text{ or } j > i \geq \pi(j) > \pi(i)\},$$

$$\text{nest } \pi = \sum_{i=1}^n \# \{j \in [n] \mid i < j < \pi(j) < \pi(i) \text{ or } i > j \geq \pi(j) > \pi(i)\}.$$

Theorem (Shin-Zeng)

For any $n \geq 1$, there is a bijection $\Phi : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ such that

$$(\text{des}, \text{fmax}, 31\text{-}2, 2\text{-}31, \text{MAD}) \pi = (\text{exc}, \text{fix}, \text{cros}, \text{nest}, \text{inv}) \Phi(\pi)$$

Let

$$\begin{aligned} A_n(x, y, q, p, s) &:= \sum_{\pi \in \mathfrak{S}_n} x^{\text{des } \pi} y^{\text{fmax } \pi} q^{(31\text{-}2) \pi} p^{(2\text{-}31) \pi} s^{\text{MAD } \pi} \\ &= \sum_{\pi \in \mathfrak{S}_n} x^{\text{exc } \pi} y^{\text{fix } \pi} q^{\text{cros } \pi} p^{\text{nest } \pi} s^{\text{inv } \pi}. \end{aligned}$$

Continued fraction due to Shin-Zeng

Theorem (Shin-Zeng2010)

$$1 + \sum_{n=1}^{\infty} A_n(x, y, q, p, s) z^n = \frac{1}{1 - b_0 z - \frac{a_0 c_1 z^2}{1 - b_1 z - \frac{a_1 c_2 z^2}{\ddots}}}, \quad (7)$$

where, for $h \geq 0$,

$$a_h = s^{2h+1} [h+1]_{q,ps}, \quad b_h = yp^h s^{2h} + (x+q)s^h [h]_{q,ps},$$

and

$$c_h = x[h]_{q,ps}, \quad [h]_{u,v} := (u^h - v^h)/(u - v).$$

Two Lemmas

Lemma

For any $n \geq 1$,

$$\begin{aligned}\mathfrak{S}_n(2-13) &= \mathfrak{S}_n(213), \mathfrak{S}_n(31-2) = \mathfrak{S}_n(312), \\ \mathfrak{S}_n(13-2) &= \mathfrak{S}_n(132), \mathfrak{S}_n(2-31) = \mathfrak{S}_n(231).\end{aligned}$$

Lemma

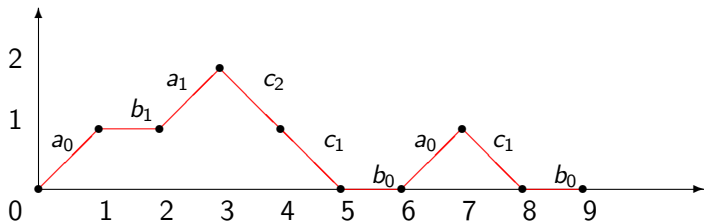
(i) A permutation $\pi \in \mathfrak{S}_n$ belongs to $\mathfrak{S}_n(321)$ if and only if $\text{nest } \pi = 0$. (ii) The mapping Φ has the property that $\Phi(\mathfrak{S}_n(231)) = \mathfrak{S}_n(321)$.

Two continued fractions of Euler, Rogers

$$\sum_{n=0}^{\infty} (n+1)!x^n = \frac{1}{1 - 2x - \frac{1 \cdot 2x^2}{1 - 4x - \frac{2 \cdot 3x^2}{\dots}}},$$

$$\sum_{n=0}^{\infty} n!x^n = \frac{1}{1 - x - \frac{1^2x^2}{1 - 3x - \frac{2^2x^2}{\dots}}}.$$

Consider the following Motzkin path γ :



The weight is $w(\gamma) = a_0^2 a_1 b_0^2 b_1 c_1^2 c_2$. Denote by $\mathfrak{M}_n$ the set of Motzkin paths of length $n \geq 1$. Then

$$1 + \sum_{n \geq 1} \sum_{\gamma \in \mathfrak{M}_n} w(\gamma) x^n = \frac{1}{1 - b_0 x - \frac{a_0 c_1 x^2}{1 - b_1 x - \frac{a_1 c_2 x^2}{\dots}}}. \quad (8)$$

Laguerre history: $h = (\gamma; p_1, \dots, p_n)$ with respect to the valuation

$$a_k = k + 1, \quad b_k = 2k + 2 \quad \text{for } k \geq 0; \quad c_k = k + 1 \quad \text{for } k \geq 1.$$

The number of Laguerre histories of length n is $(n + 1)!$.

Restricted Laguerre history: $h = (\gamma; p_1, \dots, p_n)$ with respect to the valuation

$$a_k = k + 1, \quad b_k = 2k + 1 \quad \text{for } k \geq 0; \quad c_k = k \quad \text{for } k \geq 1.$$

The number of Restricted Laguerre histories of length n is $n!$.

Two theorems due to Françon-Viennot, Foata-Zeilberger, Biane

Theorem

There are *bijections between $\mathfrak{S}_{n+1}$ and the Laguerre histories of length n .*

Theorem

There are *Bijections between $\mathfrak{S}_n$ and the Restricted Laguerre histories of length n .*

Modified Foata-Strehl action

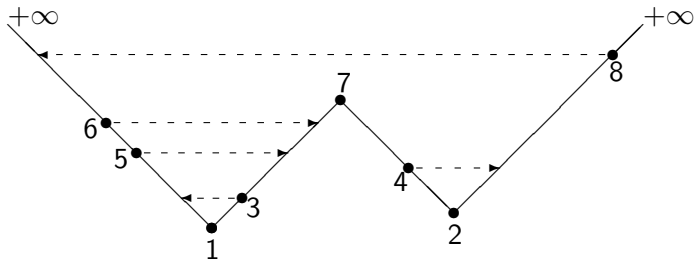


Figure: MFS-actions on 65137428

Let $\hat{\pi}$ be permutation with fixed Peaks and Valleys. Then

$$\sum_{\pi \in \text{Orb}(\hat{\pi})} t^{\text{des}(\pi)} = t^{\text{pk}(\hat{\pi})} (1+t)^{n-1-2\text{pk}(\hat{\pi})}.$$

Merci de votre attention