

SEPARATORS-A NEW STATISTIC FOR PERMUTATIONS

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SEPARATORS-DEFINITION

For $\sigma = [\sigma_1, \dots, \sigma_n] \in S_n$, we say that σ_i separates σ_{j_1} from σ_{j_2} in σ if by omitting σ_i from σ we get a new 2-block. This happens if and only if one of the following cases holds:

1. **Vertical separator**: j_1, i, j_2 are subsequent numbers and $|\sigma_{j_1} - \sigma_{j_2}| = 1$, i.e.,

$$\sigma = [\dots, \mathbf{a}, \mathbf{b}, \mathbf{a} \pm 1, \dots]$$

2. **Horizontal separator**: $\sigma_{j_1}, \sigma_i, \sigma_{j_2}$ are subsequent numbers and $|j_1 - j_2| = 1$, i.e.,

$$\sigma = [\dots, \mathbf{a}, \dots, \mathbf{a} \pm 1, \mathbf{a} \mp 1, \dots]$$

or

$$\sigma = [\dots, \mathbf{a} \pm 1, \mathbf{a} \mp 1, \dots, \mathbf{a}, \dots]$$

$Sep_V(\sigma)$ and $Sep_H(\sigma)$ are the sets of separators of σ of types vertical and horizontal respectively.

$$Sep(\sigma) = Sep_V(\sigma) \cup Sep_H(\sigma), sep(\sigma) = |Sep(\sigma)|$$

VERTICAL AND HORIZONTAL

Vertical separator and **Horizontal separator** and **separators of both types**.

$$\sigma = [1324\mathbf{6}58\mathbf{7}9]$$

$$Sep_V(\sigma) = \{3, 2, 6, 7\}, Sep_H(\sigma) = \{3, 2, 5, 8\}$$

$$Sep_V \cap Sep_H = \{3, 2\}$$

$$Sep(\sigma) = \{3, 2, 5, 6, 7, 8\}$$

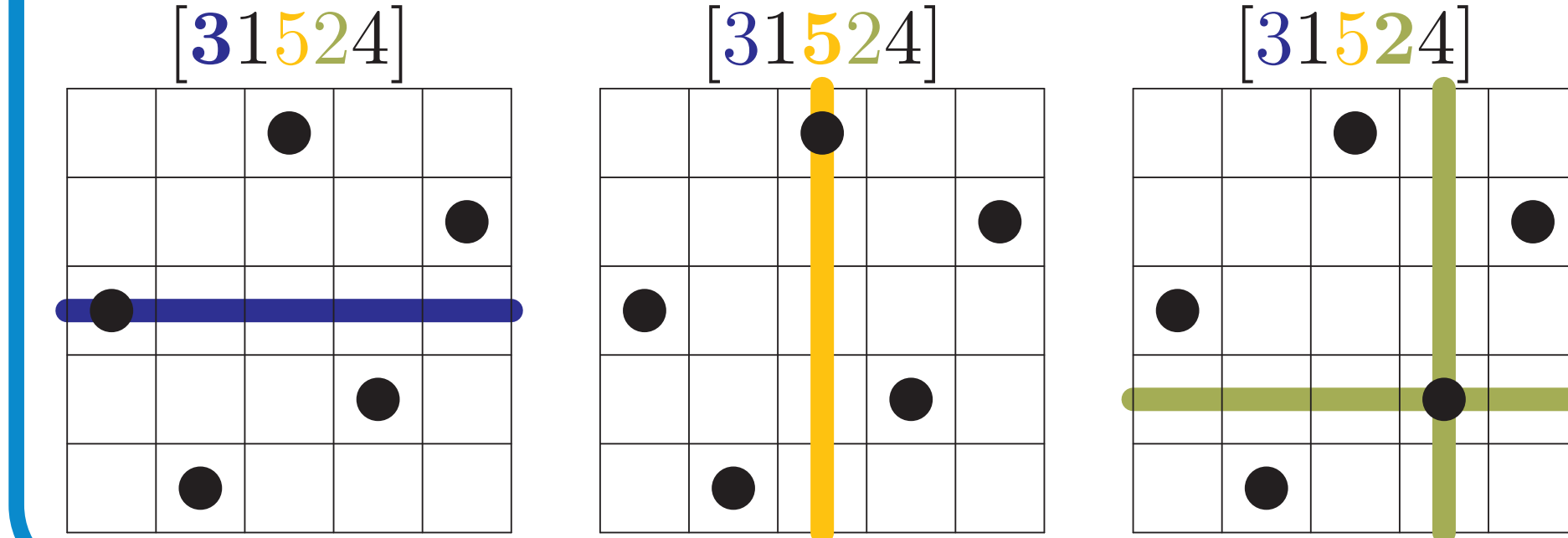
$$sep(\sigma) = |Sep(\sigma)| = 6$$



7 is a separator of type I_v , even though 7 is a part of a 2-block: 87, since by omitting 7 from σ we get a new 2-block: 78.

VERTICAL HORIZONTAL MATRIX

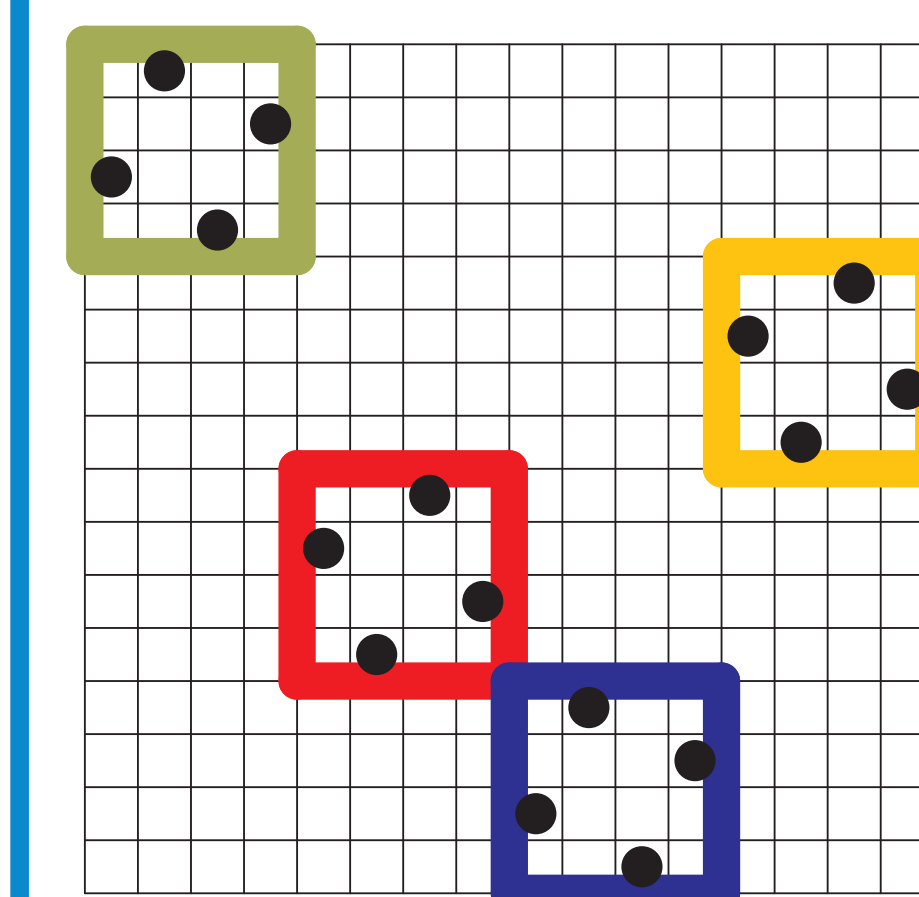
Vertical separator and **Horizontal separator** and **separators of both types**.



SEPARATORS PROPERTIES

1. Notice the significance of the word 'new' in the Definition. For example, the identity permutation has plenty of 2-blocks even though it has no separators.
2. The numbers 1 and n can only be separators of type V , while σ_1 and σ_n can only be separators of type H .
3. If σ_i is a separator of type V in σ then i is a separator of type H in σ^{-1} . Hence $Sep_V(\sigma) = Sep_H(\sigma^{-1})$
4. We have: $Sep_V(\sigma) = Sep_V(\sigma^r)$ and $Sep_H(\sigma) = Sep_H(\sigma^r)$ where σ^r is the reverse of σ .

EVERY DIGIT IS A SEPARATOR



The number of permutations in S_n which have n different separators, (i.e. every digit is a separator) is:

$$\begin{cases} 2^k k! & n = 4k \\ 0 & \text{O.W.} \end{cases}$$

COMB PERMUTATIONS

Let $\sigma = (\sigma_1, \dots, \sigma_k), \tau = (\tau_1, \dots, \tau_k)$ be two sequences such that $\{\sigma_1, \dots, \sigma_k, \tau_1, \dots, \tau_k\} = \{1, 2, \dots, 2k\}$. $\sigma \odot \tau = [\sigma_1, \tau_1, \sigma_2, \tau_2, \dots, \sigma_k, \tau_k] \in S_{2k}$.

Similarly, if $\sigma = (\sigma_1, \dots, \sigma_{k+1}), \tau = (\tau_1, \dots, \tau_k)$. For example:

$$\sigma = (1, 2, 6, 8, 9), \tau = (3, 4, 5, 7),$$

$$\sigma \odot \tau = [1, 3, 2, 4, 6, 5, 8, 7, 9].$$

When $\pi = \sigma \odot \tau$ we denote σ as π^{odd} and τ as π^{even} .

BONDS VS SEPARATORS

If $\pi_i^{odd}, \pi_{i+1}^{odd}$ is a bond of π^{odd} then the element of π^{even} which lies between them in $\pi_i^{odd}, \pi_{i+1}^{odd}$ is a vertical separator.

Example:

$\pi^{odd} = (124), \pi^{even} = (53)$. 12 is a bond in π^{odd} and thus 5 is a separator in $\pi^{odd} \odot \pi^{even} = 15234$, so we have the following:

Observation:

The number of vertical separators in $\pi^{odd} \odot \pi^{even}$ is equal to the total number of bonds in π^{odd} and in π^{even} .

THE GENERATING FUNCTION

$$h(z, v) = \sum_{m, n} a_{m, n} z^m v^n = \sum_{m_o, m_e \geq 0} (m_o + m_e)! (z^2 + \frac{2z^4(v-1)}{1-z^2(v-1)})^{m_o} * (z^2 + \frac{2z^4(v-1)}{1-z^2(v-1)})^{m_e} + \sum_{m_o, m_e \geq 0} (m_o + m_e)! ((z^2 + \frac{2z^4(v-1)}{1-z^2(v-1)})^{m_o} \frac{1}{z}) * ((z^2 + \frac{2z^4(v-1)}{1-z^2(v-1)})^{m_e} z)$$

Where $a_{n, m}$ the number of permutations of S_n with m bonds and where $*$ is the Hadamard product in $\mathbb{Q}[[u]][[z]]$.

EXPECTATION OF SEPARATORS

- The expectation of the number of vertical (or horizontal) separators in a randomly chosen n -permutation is $\frac{2(n-2)}{n}$. The asymptotic value of the expectation is 2.
- The expectation of the number of separators in a randomly chosen n -permutation is

$$\frac{4(n^3 - 6n^2 + 14n - 13)}{n(n+1)(n+2)}$$

The asymptotic value of the expectation is 4.

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