



On the poset of non-attacking-king permutations

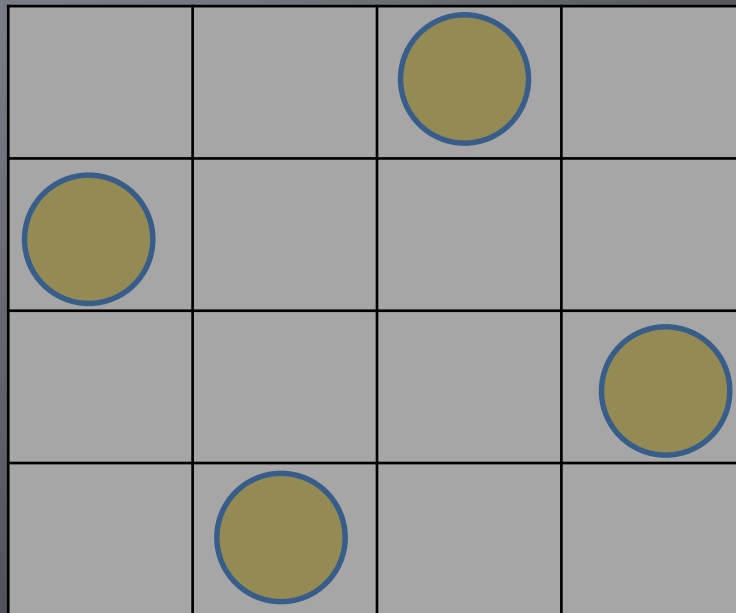
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Joint work with E.Bagno and S.Reches

Non Attacking Kings

A non-attacking-king permutation is a permutation $\pi \in S_n$ such that

$$|\pi_{i-1} - \pi_i| \neq 1 \text{ for each } i \in \{2, \dots, n\}$$





King permutations

- K_n is the set of all 'king' permutations in S_n
- $K_2 = K_3 = \phi$
- $K_4 = \{[2413], [3142]\}$
- K_n is closed to the reverse and inverse actions.



Enumerative data

- An explicit formula for the number of king permutations was given by Robbins.
He also showed that when n tends to infinity, the probability of picking such a permutation from S_n approaches e^{-2}
- This set is counted in OEIS A002464.

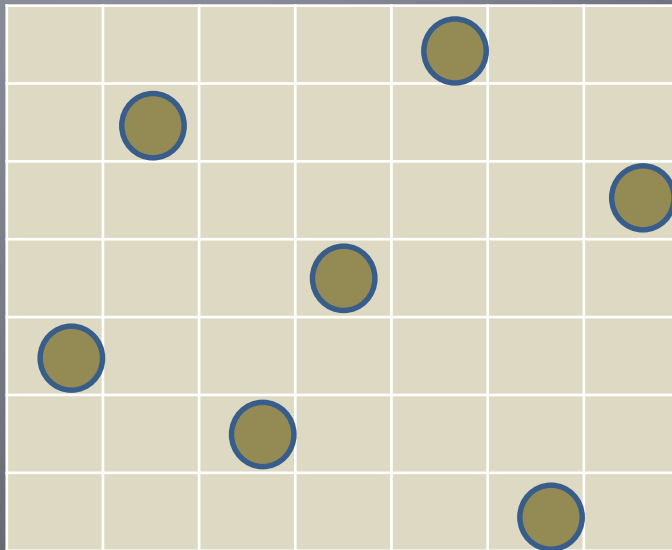


The poset of king permutations

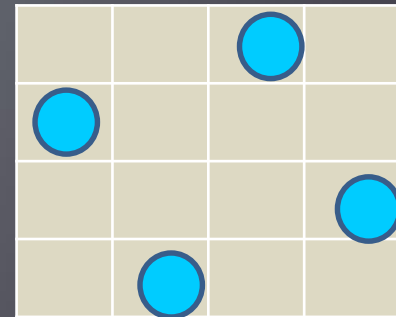
- Let $\sigma, \pi \in \bigcup_{n \in \mathbb{N}} S_n$. We say that σ **contains** π if there is a sub-sequence of elements of σ that is order-isomorphic to π
- If σ contains π , then we write $\pi \preceq \sigma$.
If $\pi \preceq \sigma$ but $\pi \neq \sigma$ then we can write $\pi < \sigma$.

[3624715] contains [3142] as the sub-sequences 6275 and 6475

[3624715]

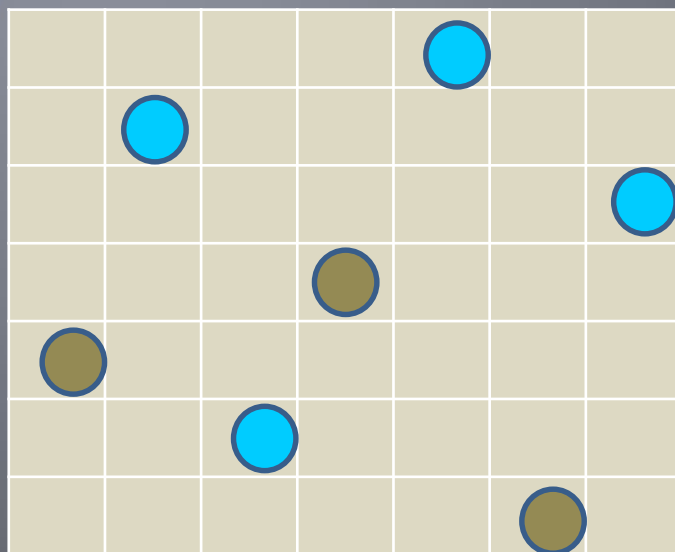


[3142]

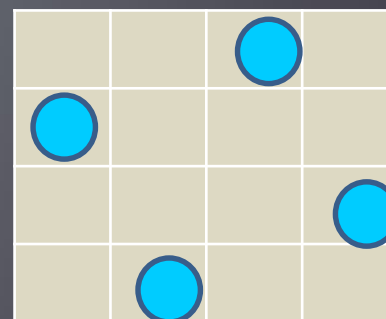


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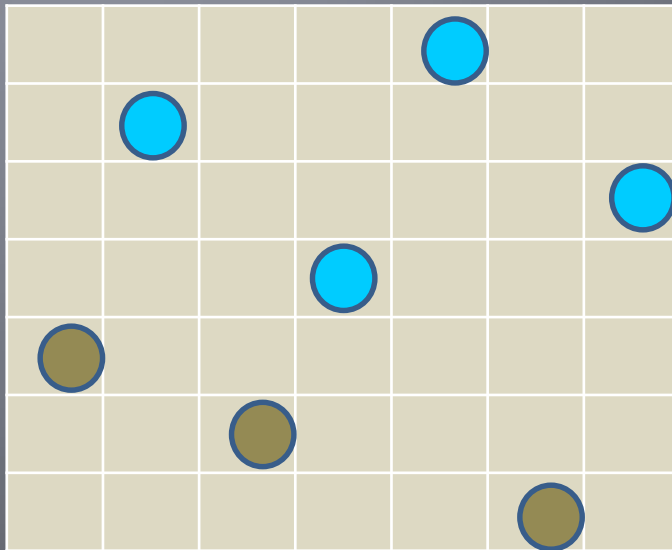


[3142]

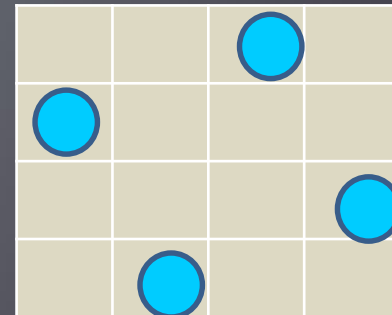


[3624715] contains [3142] as the sub-sequences 6275 and 6475

[3624715]



[3142]



Princes

Let $\sigma \in K_n$ and let $\pi < \sigma$, if $\pi \in K_{n-1}$, then π is called a prince.





Some interesting questions about the princes

- Are there kings without princes?
- If there are, what can be said about their structure?
- Are there kings in K_n with n princes?
- If so, what is their structure?



Example

Which digits of the permutation

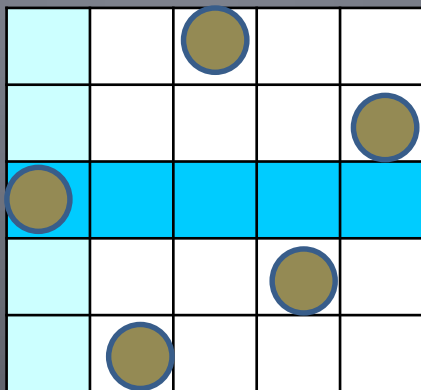
[31524]

can be removed in order to get a prince?

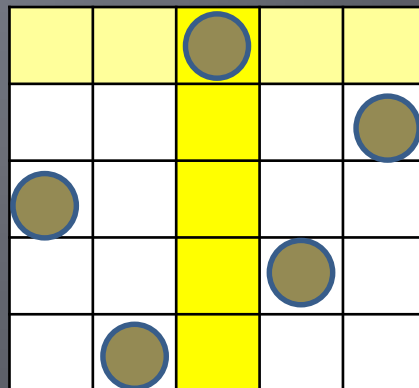
Vertical separator and Horizontal separator and separators of both types

[31524]

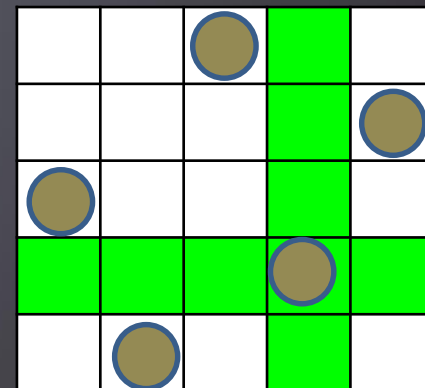
[31524]



[31524]



[31524]





SEPARATORS

For $\sigma \in S_n$ we say that the digit ***a*** is a separator if by omitting ***a*** we get a ***new 2-block***.

This happens if and only if one of the following cases holds:

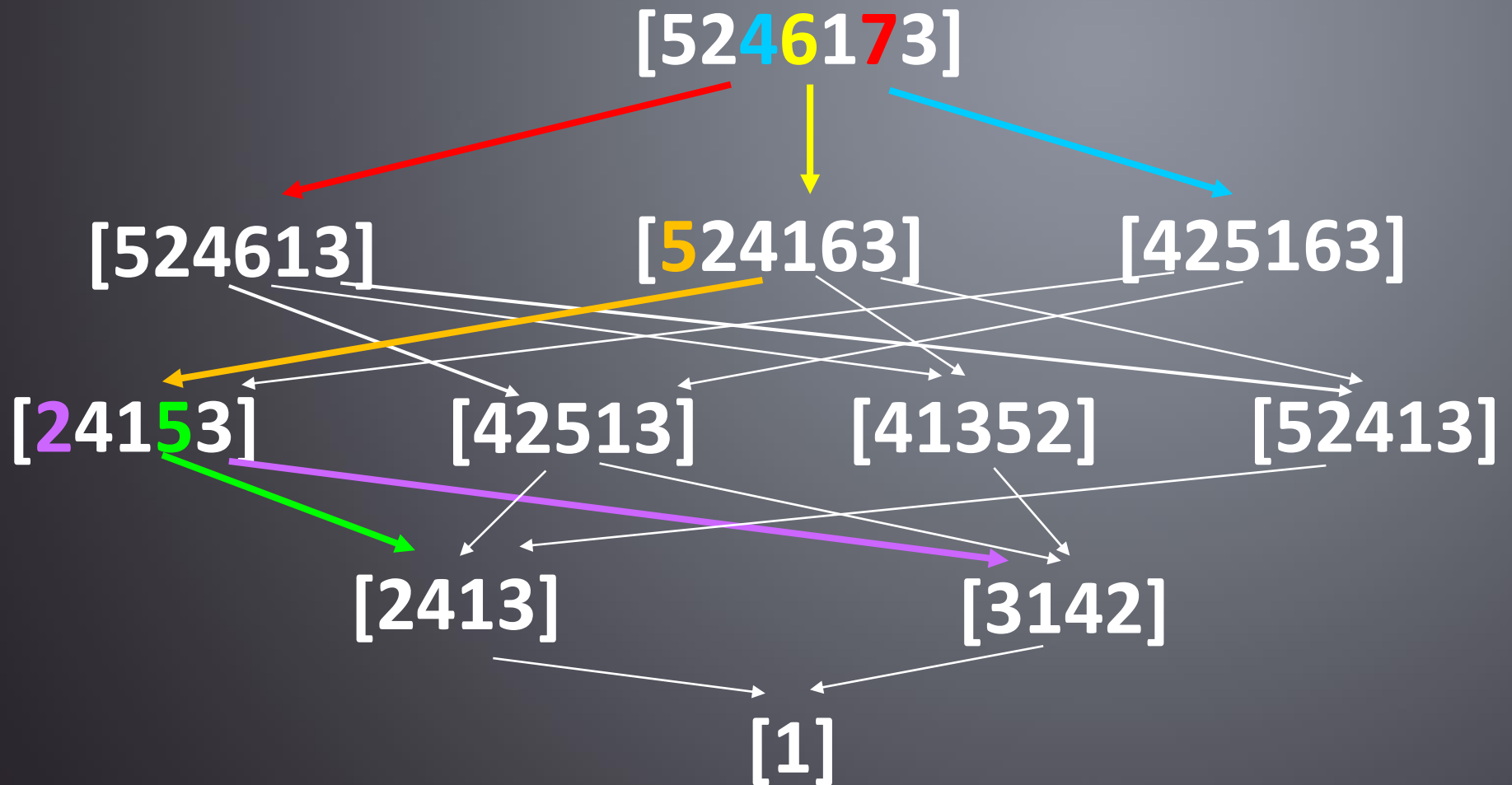
- Vertical separator:

$$\sigma = [\dots, b, \mathbf{a}, b \pm 1, \dots]$$

- Horizontal separator:

$$\sigma = [\dots, \mathbf{a}, \dots, a \pm 1, a \mp 1, \dots]$$

The downset of $[5246173]$





Buildings Blocks Theorem

For every $\pi \in K_n$ ($n \geq 4$),
either $[3142] \preceq \pi$ or $[2413] \preceq \pi$



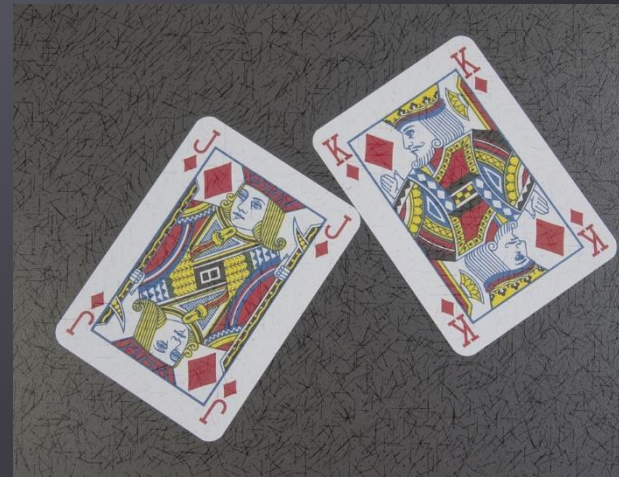
Theorem:

If you have a grandson king you also have a prince son.

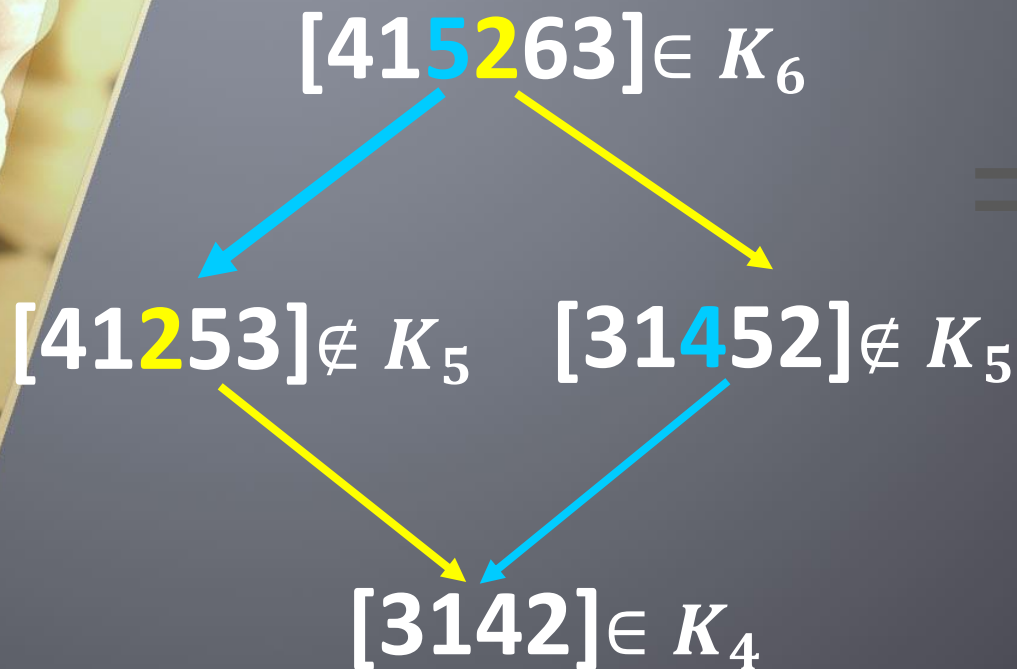
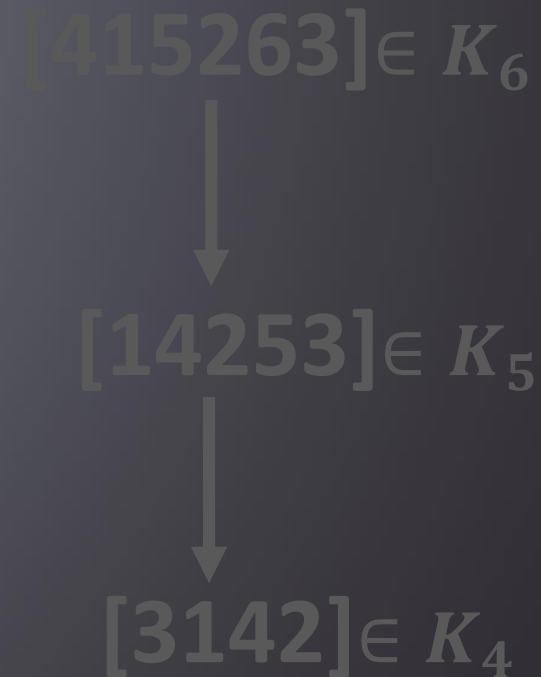
Let $\sigma \in K_n$ with $n > 4$, and let $\pi \in K_{n-2}$ be such that $\pi < \sigma$.

Then there exists $\tau \in K_{n-1}$ such that

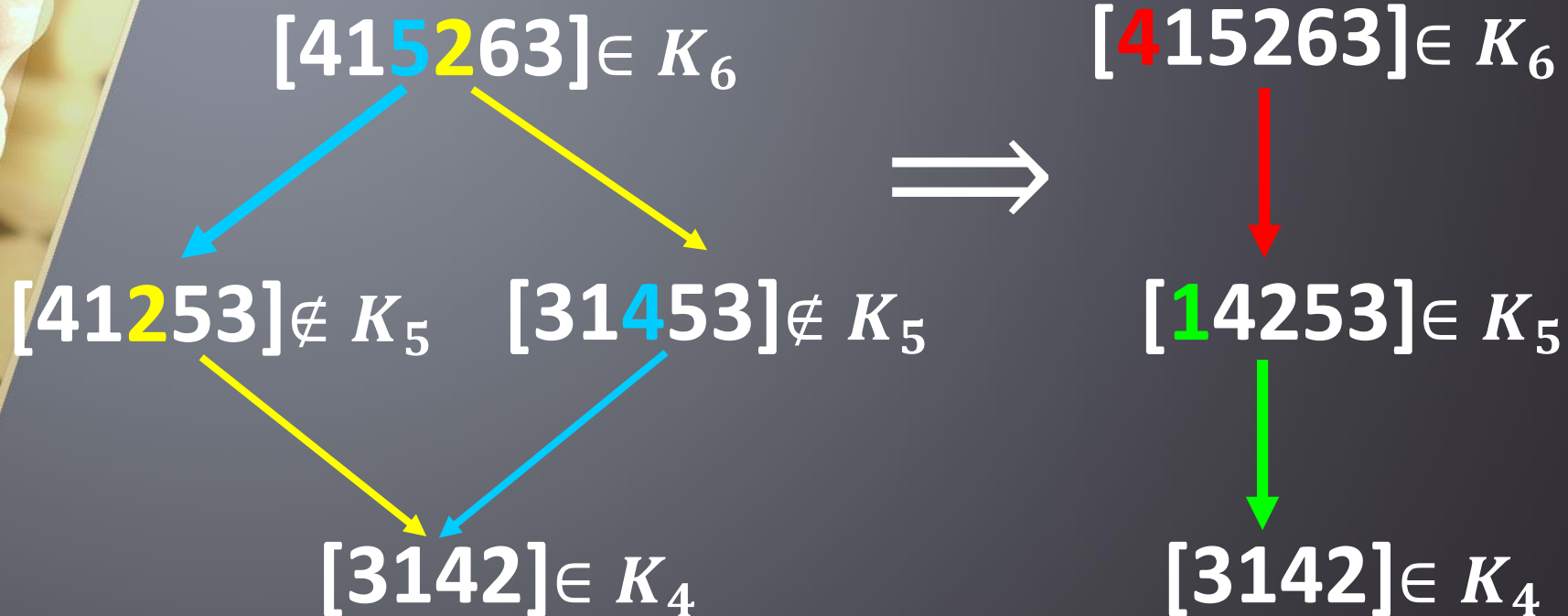
$$\pi < \tau < \sigma .$$



Example: grandson $\Rightarrow$ prince



Example: grandson $\Rightarrow$ prince





Chain of generations

For each two king permutations $\pi < \sigma$ there exists a chain of king permutations

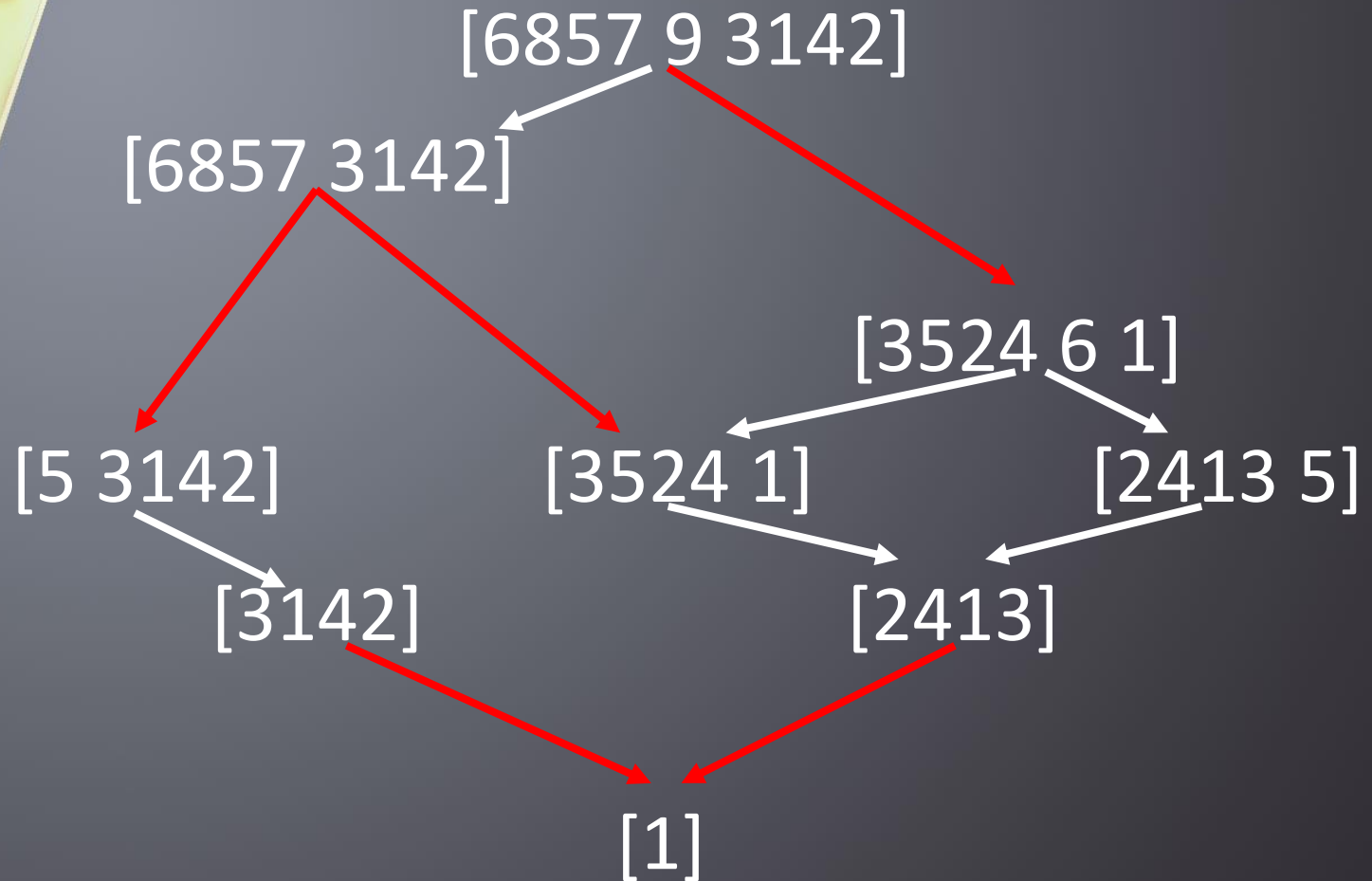
$$\pi = \pi^0 < \pi^1 < \dots < \pi^k = \sigma$$

such that $|\pi^i - \pi^{i-1}| \in \{1, 3\}$.



[6857 9 3142]

Chain of generations



$$|\pi^i - \pi^{i-1}| \in \{1, \mathbf{3}\}$$



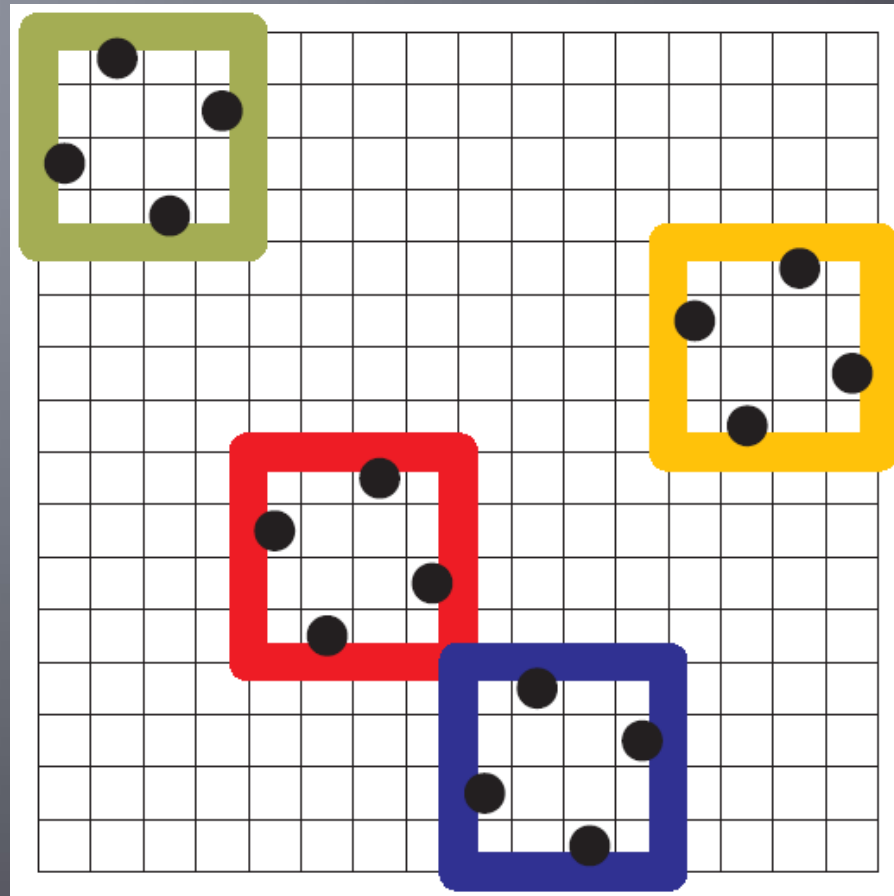
Kings without princes – Theorem

The following conditions are equivalent for each $\pi \in k_n$ with $n \geq 4$:

- π has no princes.
- For each $i \in \{1, \dots, n\}$ by removing i from π we get a block of length 3.
- There are $\pi^1, \dots, \pi^k \in \{[3142], [2413]\}$ and $\pi' \in S_k$ such that $\pi = \pi'[\pi^1, \dots, \pi^k]$.

King without princes – Example

[14,16,13,15, 7,5,8,6, 2,4,1,3, 11,9,12,10]
=4213[[2413] , [3142], [2413], [3142]]





The number of permutations in K_n which have no princes.

The number of permutations in K_n which have no princes:

$$\begin{cases} k! 2^k & n = 4k \\ 0 & O.W. \end{cases}$$



Open (and closed) questions

- ✓ Are there kings without princes?
- ✓ If there are, what can be said about their structure?
- ✓ Are there kings in K_n with n princes?
- ? If so, what is their structure?



Thanks for your attention



The full paper:

