Enumerative combinatorics of intervals in the Dyck pattern poset

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- The number of elements and covering relations in the interval $[UD, P]$ for very specific classes of Dyck paths P.

The Dyck Pattern Poset

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Definition

Given two Dyck paths P and Q, we say that P is a pattern in Q and we write $P \leq Q$ when P is a (non necessarily consecutive) subword of Q. Any sequence of positions in Q whose corresponding subword equals P is called an occurrence of P in Q. The resulting poset is called the Dyck pattern poset and it will be denoted by $\mathcal{D}$.

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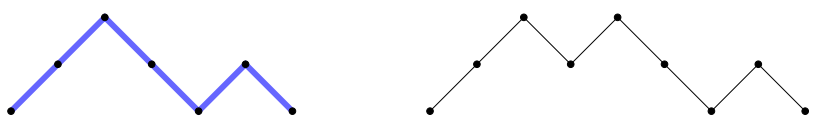

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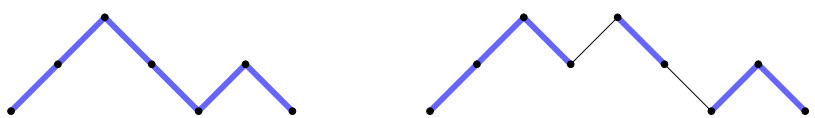

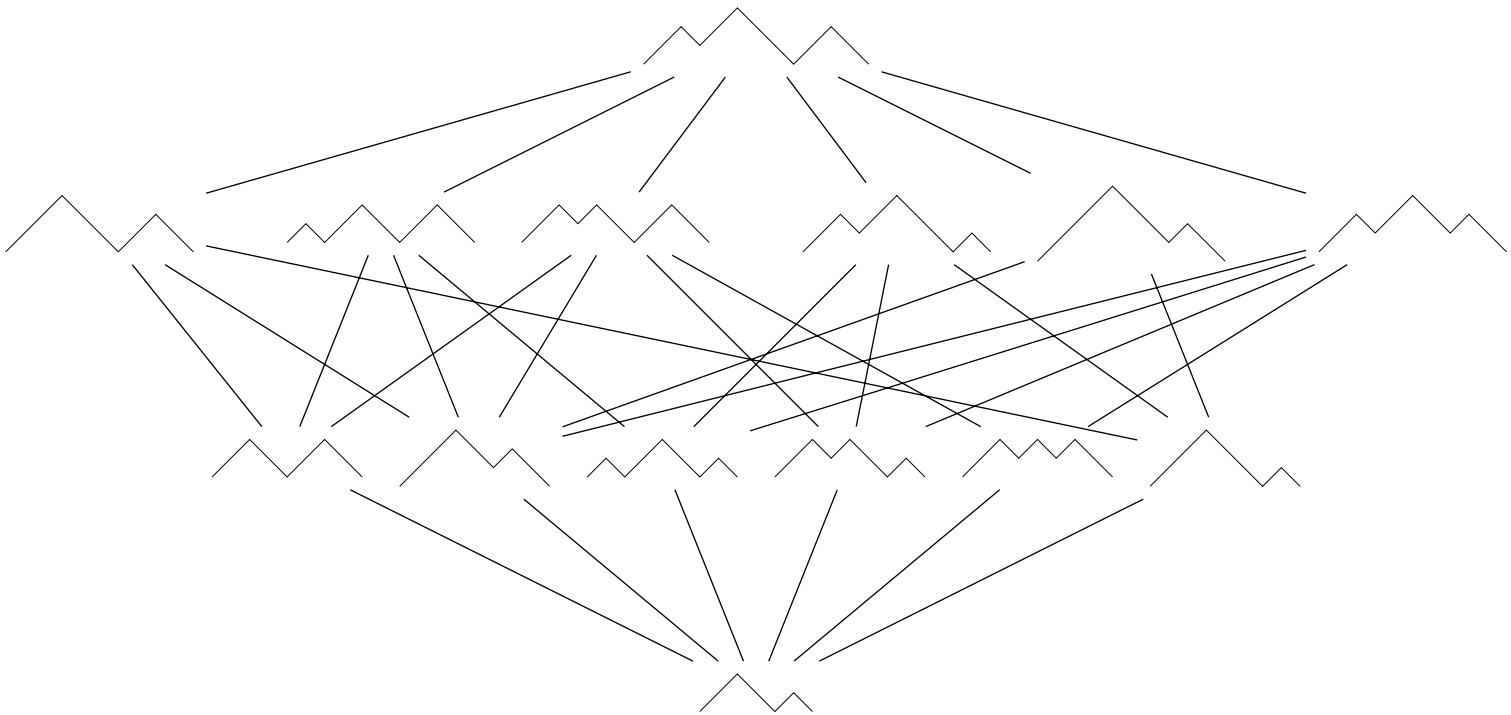

Figure: The Hasse diagram of the interval [UUDDUD,UUDUUDDDUUDD] in the Dyck pattern poset.

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- For a general Dyck path P, a recursive method to find the generating function of the class of $P-$avoiders is known and this generating function turns out to be rational.
- For a general Dyck path P, an asymptotic formula for the number of $P-$avoiders is known.

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Goal of the Talk

In this talk we will be concerned with the study of $s_0[UD, P]$ and $s_1[UD, P]$ for some specific classes of Dyck paths P.

The Size of the Interval $[UD, (UD)^n]$

Our first goal is to find an explicit formula for the number of elements of the interval $[UD, (UD)^n]$ where $n \geq 0$.

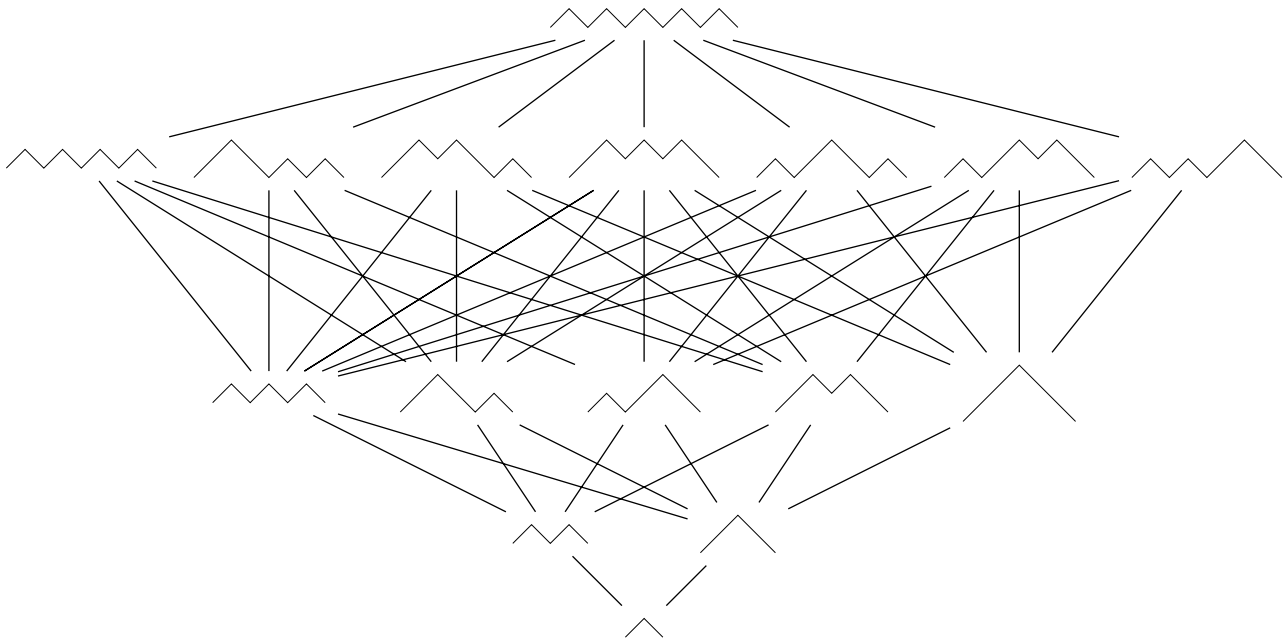

Figure: The Hasse diagram of the interval $[UD, (UD)^5]$ in the Dyck pattern poset, thus $s_0[UD, (UD)^5] = 16$.

A characterization of the patterns in $(UD)^n$

For a Dyck path P denote with $\mathrm{asc}(P)$ the number of ascents of P (where an *ascent* of a Dyck path is a maximal consecutive substring of P of the form U^m for some $m \geq 1$), which is also the number of peaks of P. The next lemma characterizes the patterns P in $(UD)^n$ in terms of $\mathrm{asc}(P)$.

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Let $n \geq 1$, $1 \leq k \leq n$ and P be a Dyck path of semilength k. Then $P \leq (UD)^n$ if and only if $\mathrm{asc}(P) \geq 2k - n$.

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Example

According to the previous criterion $P = U^3D^2UD^2UD \leq (UD)^7$ because P has semilength 5, furthermore $\mathrm{asc}(P) = 7$, therefore $\mathrm{asc}(P) \geq 2 \cdot 5 - 7$.

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- Continue this process as long as you can.

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- Since P begins with 3 U steps, choose the first 3 U steps available in $(UD)^7$.

- Since P continues with 2 D steps, choose the first 2 D steps available in $(UD)^7$ after the previous choice.

- Continue this process as long as you can.
- Convince yourself that this process terminates running out of all the letters of P if and only if $P \leq (UD)^7$ and that this is the case precisely when $\mathrm{asc}(P) \geq 2k - n$.

Recall that, for $n \geq 1$ and $1 \leq k \leq n$, the Dyck paths of semilength n and having k ascents are counted by the *Narayana number* $N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. Therefore we obtain the following formulas.

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Proposition

Suppose $n \geq 1$ and $1 \leq k \leq n$, then

(i)

$$s_0^{(k)}[UD, (UD)^n] = \sum_{m=\max\{1,2k-n\}}^{k} N_{k,m}$$

(ii)

$$s_0[UD, (UD)^n] = \sum_{k=1}^{n} \sum_{m=\max\{1,2k-n\}}^{k} N_{k,m}$$

Sequence $(s_0^{(k)}[UD, (UD)^n])_{n \geq k \geq 0}$ appears as A137940 in OEIS. The first few lines of the associated triangle are recorded below:

$n \backslash k$	1	2	3	4	5	6	7	8	9
1	1								
2	1	1							
3	1	2	1						
4	1	2	4	1					
5	1	2	5	7	1				
6	1	2	5	13	11	1			
7	1	2	5	14	31	16	1		
8	1	2	5	14	41	66	22	1	
9	1	2	5	14	42	116	127	29	1

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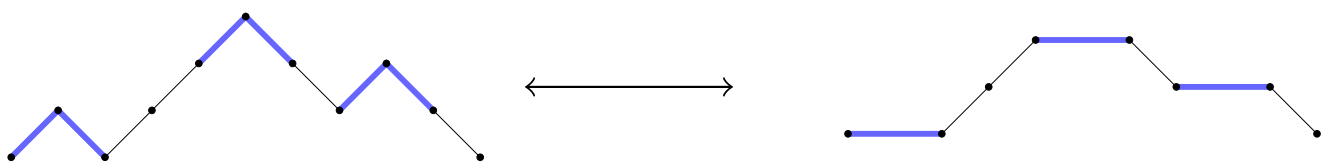
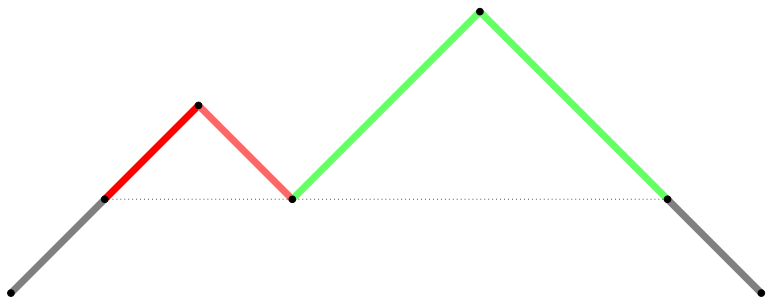


The size of the interval $[UD, U^{a+h}D^aU^bD^{b+h}]$

For $1 \leq a \leq b$ and $h \geq 1$ denote by $Q_{a,b}^{(h)}$ the generic Dyck path with two peaks $U^{a+h}D^aU^bD^{b+h}$.



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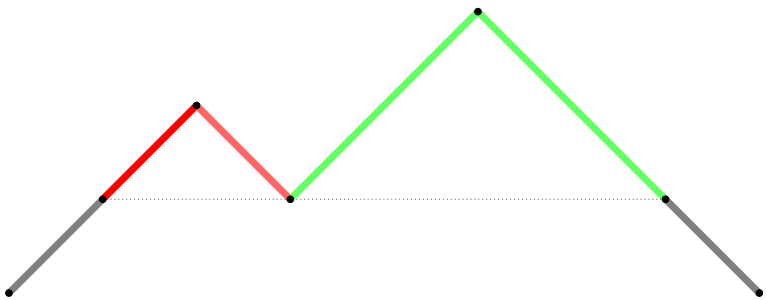

Our next goal is to find an explicit formula for the number of elements in the interval $[UD, Q_{a,b}^{(h)}]$.

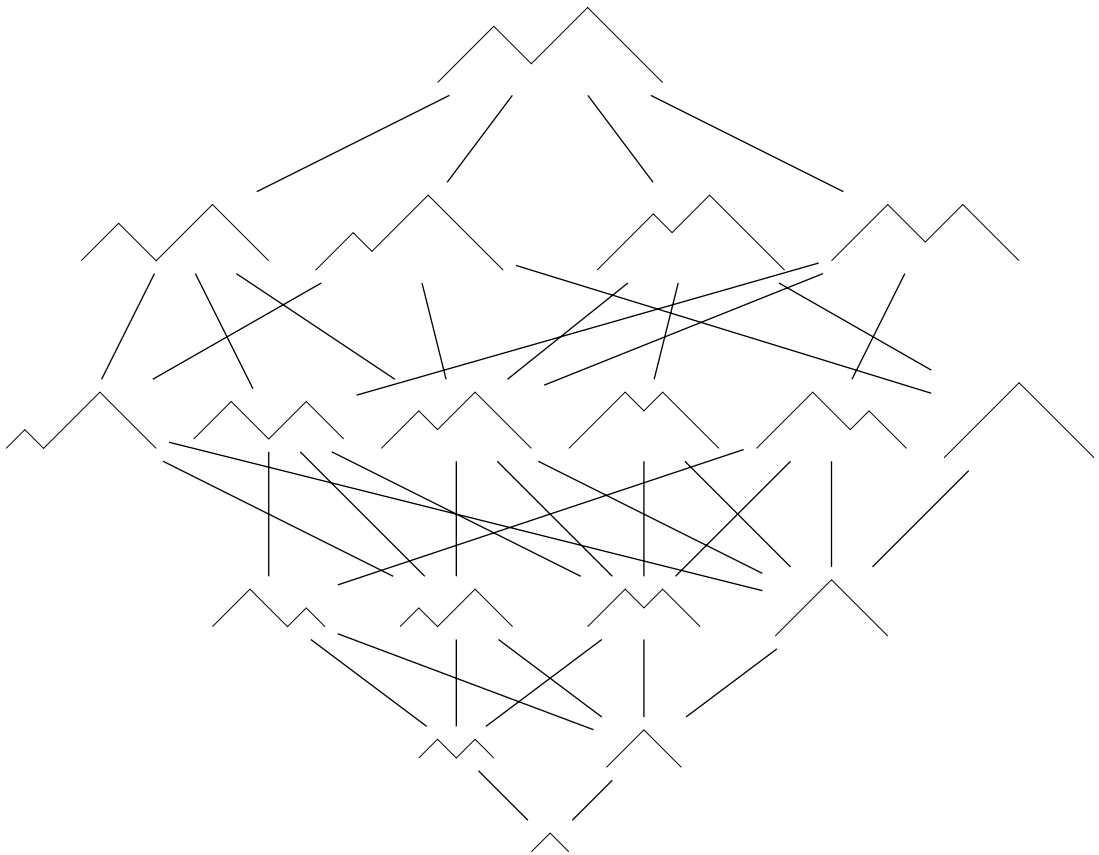

Figure: The Hasse diagram of the interval $[UD, Q_{2,3}^{(1)}]$ in the Dyck pattern poset, thus $s_0[UD, Q_{2,3}^{(1)}] = 18$.

Denote by $\varphi_h(a, b)$ the number of Dyck paths having exactly two peaks in the interval $[UD, Q_{a,b}^{(h)}]$.

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Let $1 \leq a \leq b$, then

$$\varphi_0(a, b) = \sum_{k=1}^{a-1} (a - k)(b - k) = \frac{a(a+1)(3b - a + 1)}{6}.$$

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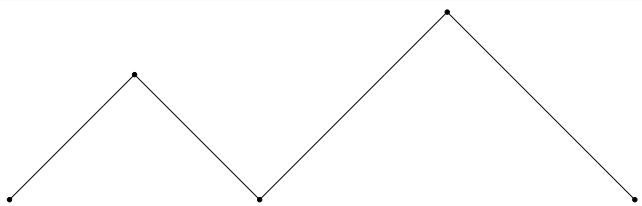

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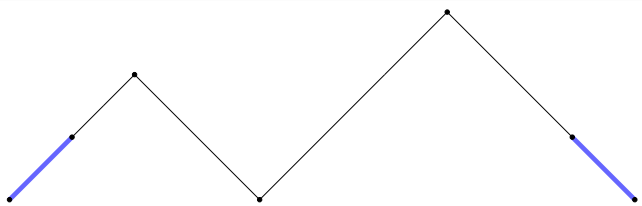

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Let $1 \leq a \leq b$, then

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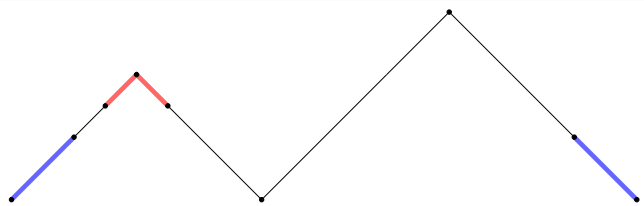

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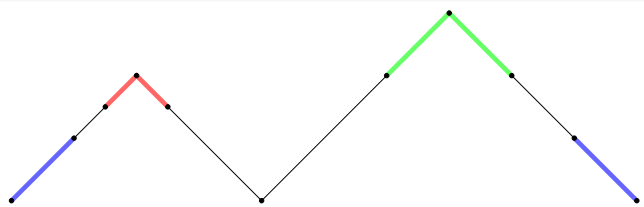

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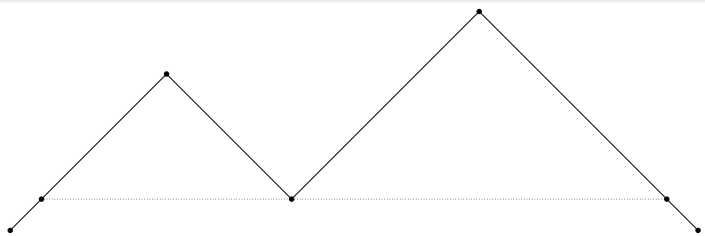

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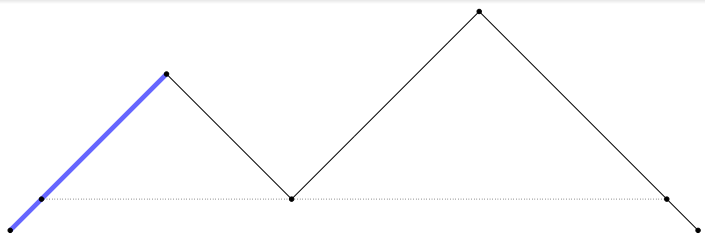

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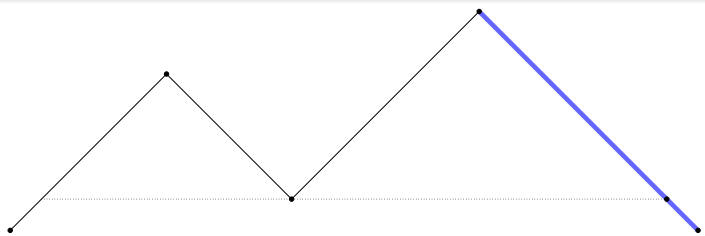

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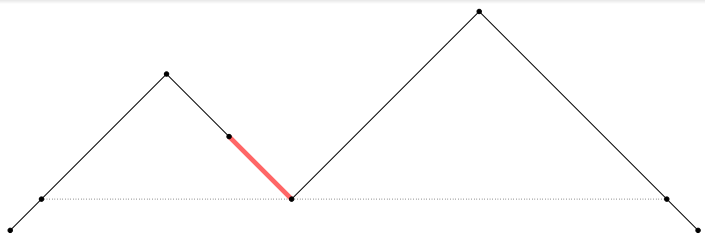

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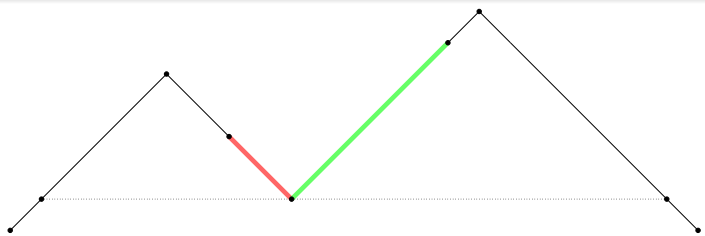

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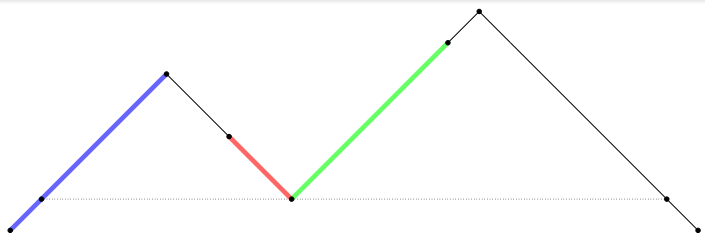

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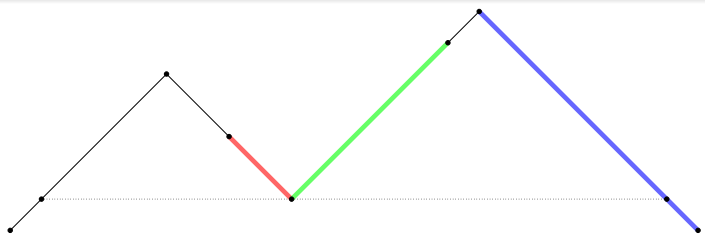

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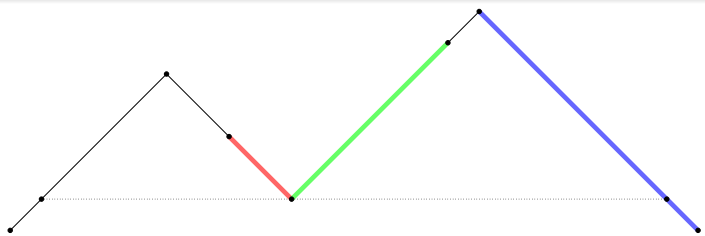

Proposition

Let $1 \leq a \leq b$ *and* $h \geq 0$, *then*

$$s_0[UD, Q_{a,b}^{(h)}] = \frac{a(a+1)(3b-a+1)}{6} + hab + b.$$

The previous results can be refined by counting the number of elements in the interval $[UD, Q_{a,b}^{(h)}]$ with semilength r.

Proposition

Let $1 \leq a \leq b$, $h \geq 0$ and $2 \leq r \leq a + b + h$, then

$$\varphi_h^{(r)}(a,b) = \sum_{i=\max\{1,r-b-h\}}^{\min\{a,r-1\}} \min\{r-i, b\} - \max\{a, r-a-h-1\}$$

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$$= \begin{cases} \binom{r}{2} & 2 \leq r \leq a+1 \\ ab - a(r-a-h-1) - \binom{a+b-r+1}{2} + \binom{b+1-r}{2} & a < b,\ a+2 \leq r \leq b+h \\ (a+b+h-r+1)^2 - \binom{a+b-r+1}{2} & b+h+1 \leq r \leq a+b+h \end{cases}$$

Remark

The triangular array determined by $(\varphi_0(a,b))_{1\leq a\leq b}$ is sequence A082652 in OEIS, which counts the numbers of squares that can be found in an $a \times b$ rectangular grid.

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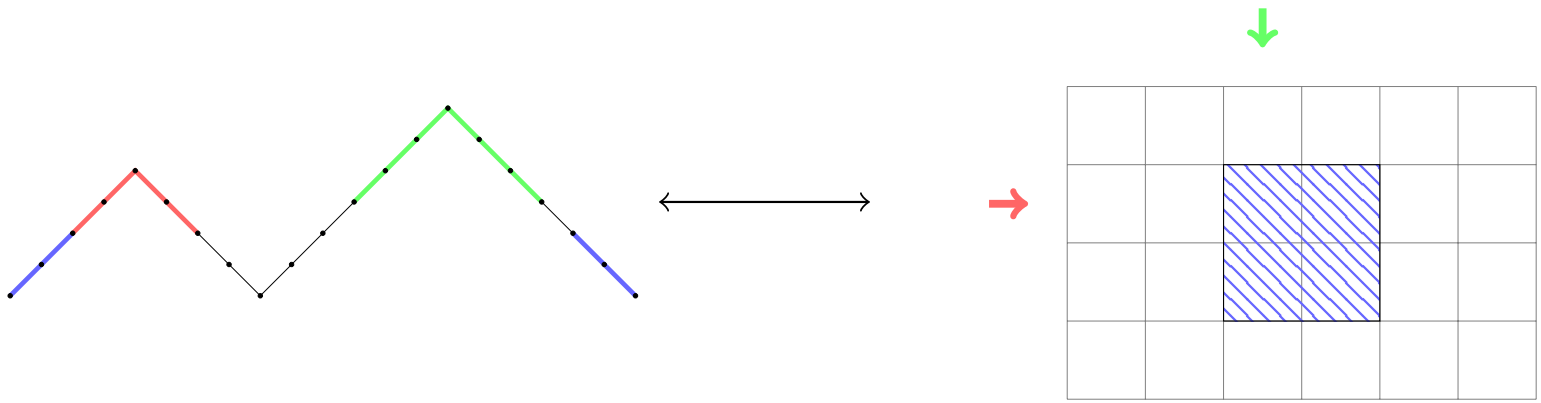

$$Q_{i,j}^{(k)} \longleftrightarrow [i, i+k] \times [j, j+k]$$

Number of edges in the Hasse diagram of the interval $[UD, U^aD^aU^bD^b]$

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Lemma

Let $(i,j,k) \in \mathbb{N}^3$, then the number of Dyck paths covered by $Q^{(k)}_{i,j}$ is

$$\begin{cases} 0 & (i,j,k) \in \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\} \\ 1 & (i,j,k) \in \{(1,1,0),(i,0,k),(0,j,k) : \ i+k, j+k \geq 2\} \\ 2 & (i,j,k) \in \{(i,1,0),(1,j,0),(1,1,k) : \ i,j \geq 2, \ k \geq 1\} \\ 3 & (i,j,k) \in \{(i,j,0),(i,1,k),(1,j,k) : \ i,j \geq 2, \ k \geq 1\} \\ 4 & (i,j,k) \in \{(i,j,k) : \ i,j \geq 2, \ k \geq 1\} \end{cases}$$

Now we can count the edges in the Hasse diagram of the interval $[UD, Q_{a,b}^{(0)}]$ as follows

$$s_1[UD, Q_{a,b}^{(0)}] = \sum_{n \geq 0} \Delta_n \cdot n$$

where Δ_n denotes the number of Dyck paths in the interval $[UD, Q_{a,b}^{(0)}]$ covering exactly n Dyck paths.

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Proposition

Let $1 \leq a \leq b$, then

$$s_1[UD, Q_{a,b}^{(0)}] = -\frac{1}{3}(2a^3 - 6a^2b + a - 3b + 3).$$

Final Remarks

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- Very little is known about the Moebius function of the Dyck pattern poset.
- This class of problems does not seem to have been investigated yet in the case of the permutation pattern poset.

Thank you!