

Countable universal and existentially closed permutations in geometric grid classes

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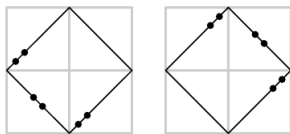
Permutation Patterns Meeting
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OVERVIEW

- 1 Introduction
- 2 EC permutations
- 3 Extensions
- 4 Questions

PERMUTATION DEFINITIONS

- We view permutations as structures in a language of two linear orders, and so embeddings correspond to containment.
- This corresponds to the usual picture of permutations as points in the plane, with $<_1$ giving the horizontal order and $<_2$ the vertical.
- A *geometric grid class* [1] consists of permutations placeable on a given grid of cells, each either empty or containing an increasing or decreasing diagonal.



- A *monotone grid class* is similar, except in the non-empty cells, we only demand increasing or decreasing.

COUNTABLE UNIVERSALITY

- Consider the permutation class $\mathcal{C} = Av(\mathcal{P})$.
- $\mathcal{C}$ has a *finite universal permutation* if there is some countable permutation avoiding the patterns in $\mathcal{P}$ and embedding all finite elements of $\mathcal{C}$.
- This is equivalent to atomicity/the joint embedding property.
- $\mathcal{C}$ has a *countable universal permutation* if as above except embedding all countable elements of $\mathcal{C}$.

EXISTENTIAL CLOSEDNESS

Definition

A structure $M \in \text{Av}(\mathcal{P})$ is *existentially closed* (EC) if given

- 1 $A \subset M$ finite
- 2 A disjoint set of points B with the isomorphism type of $A \cup B$ specified

If M can be extended to include B while staying in $\text{Av}(\mathcal{P})$, then a configuration isomorphic to B is already realized over A in M . We will always assume M is countable.

Example

Consider the class of linear orders. An EC linear order is without endpoints ($a \in M, b < a$ or $b > a$) and dense ($a_1, a_2 \in M, a_1 < b < a_2$).

The unique such countable linear order is $\mathbb{Q}$.

EXISTENTIAL CLOSEDNESS (CONT.)

- It is particularly nice when the class of EC structures contains a unique countable member (e.g. linear orders).
- Every member of $Av(\mathcal{P})$ can be extended to an EC one.
- Thus, if there is a unique countable EC member, it provides a canonical countable universal permutation.

COUNTABLE UNIVERSAL PERMUTATIONS

Theorem

Geometric grid classes have countable universal permutations.

- Idea: Make every line a copy of $\mathbb{Q}$, and make everything simultaneously dense in everything else (see board).

UNIQUE EC PERMUTATIONS

Question

When does a geometric grid class have a unique countable EC permutation?

- The countable universal structure from the previous slide is a natural candidate.
- But it would require the grid to be minimal (see board).

Conjecture

For a geometric grid class, the previous construction with a minimal grid gives the unique countable EC permutation.

- Hutschenreiter [3] looked at avoiding patterns of length 3.
- All such atomic classes have a unique countable EC permutation, except $Av(321)$ and $Av(231)$ are open.
- All except possibly $Av(231)$ have a (possibly infinite) geometric gridding.

PROVING A UNIQUE EC PERMUTATION

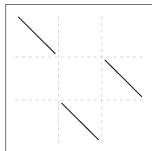
Theorem (Albert, Burris [2])

To prove the existence of a unique countable EC structure, it is enough to understand obstructions to amalgamation.

- Expectation: Expanding the language by unary predicates naming the cells and adding some binary relations encoding the cuts a point defines in other cells, the class should have amalgamation.
- So we need to show these extra relations are existentially definable in the minimal grid (but the binary relations are easy).

EXISTENTIAL DEFINABILITY EXAMPLE

- Consider $Av(123, 231)$.



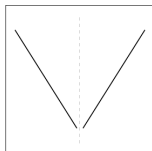
- Let C_1 be the bottom filled cell, C_2 the middle filled cell, and C_3 the top filled cell.
 - For each cell, we need to find a finite pattern guaranteeing a point is in that cell.
- ① $x \in C_1$ if there is some y such that $y >_1 x$ and $y >_2 x$.
 - ② $x \in C_2$ if there is some y such that $y <_1 x$ and $y <_2 x$.
 - ③ $x \in C_3$ if there are y_1, y_2 such that $x <_1 y_1 <_1 y_2$ and $y_1 <_2 y_2 <_2 x$.

UNDERSTANDING BASES

- For geometric grid classes, the basis is known to be finite but there is no algorithm for finding it. (For monotone, finiteness is open).
 - Understanding existential definitions for the cells should also make the forbidden patterns clear.
 - In the expanded language, there should be an explicit description of the forbidden substructures.
 - Existential definitions would translate these back to forbidden permutations.
 - The forbidden structures say things like
- 1 No point can be in two different cells.
 - 2 Cells are monotone in proper direction.
 - 3 No points are in empty cells.

BASIS EXAMPLE

- Consider $Av(132, 231)$.



- 1 $Left(x)$ if there is some y below and right of x .
 - 2 $Right(x)$ if there is some y below and left of x .
- From the condition that no point is in distinct cells, we see we cannot have a point x with y_1 below and right of it and y_2 below and left of it.
 - Thus we must forbid 132 and 231.

UNDERSTANDING BASES (CONT.)

- We only need a single existential definition for a unique countable EC structure.
- But a single cell might have many existential definitions.
- To understand the basis for the class, we would need to find a basis of definitions for each cell.

MONOTONE GRID CLASSES

- From a couple examples, it seems that most of this could also work for monotone grid classes.
- Don't always have atomicity (it is decidable when), so won't always have a unique countable EC structure.
- Can still hope for existential definability of the cells, and thus an understanding of the bases.
- Model-theoretically, might hope to always have a model companion.

INFINITE GRIDS

- This may also go through for infinite grids.
- A simple case seems to be when the grid is an infinite direct/skew sum of finite grids.
- In this case, just take $\mathbb{Q}$ -many copies of the finite grid.
- No obvious obstruction to more general infinite grids.
- Already, $Av(321)$ (the staircase grid) was left open by Hutschenreiter.

QUESTIONS

Question

Are the cells of a geometric/monotone grid class existentially definable in the minimal grid? (First-order definability might be enough.)

Question

What can be said about basis of existential definitions a given cell? Is it finite? Are there bounds on the length of its elements?

REFERENCES

- [1] Michael Albert, M Atkinson, Mathilde Bouvel, Nik Ruškuc, and Vincent Vatter, *Geometric grid classes of permutations*, Transactions of the American Mathematical Society **365** (2013), no. 11, 5859–5881.
- [2] Michael Albert and Stanley Burris, *Bounded obstructions, model companions and amalgamation bases*, Mathematical Logic Quarterly **34** (1988), no. 2, 109–115.
- [3] Lisa Hutschenreiter, *The Model Companions of Permutation Pattern Avoidance Classes*, M.Sc. Thesis, TU Dresden, 2015.