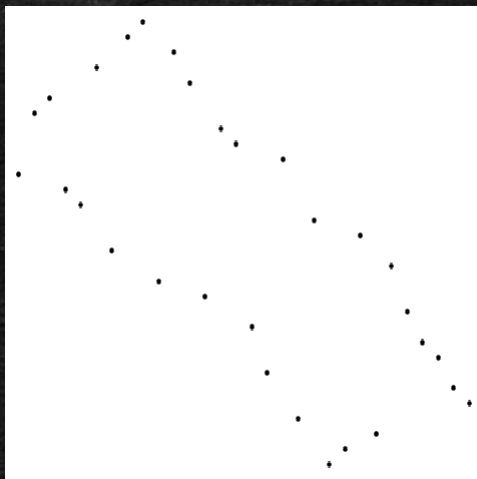
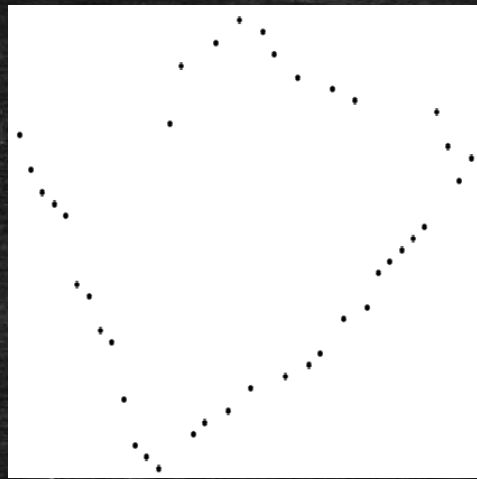
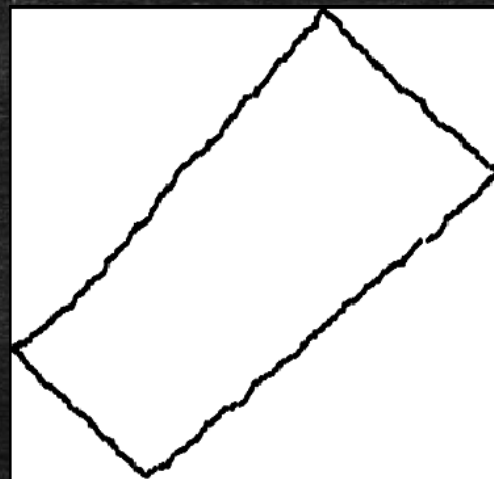
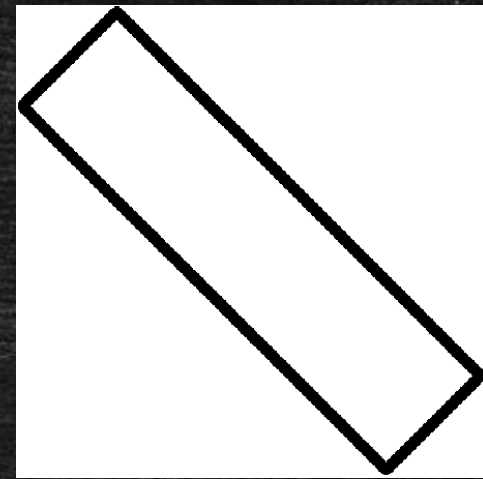




SQUARE PERMUTATIONS ARE
TYPICALLY RECTANGULAR

(joint work with E. Slivken)

J. BORGA, UZH

JUNE 20th, 2019

ENUMERATION:

$$\begin{aligned} \# \text{ square permutations of size } n &= 2(n+2)4^{n-3} + \underbrace{4(2n-5)\binom{2n-6}{n-3}}_{o(n4^{n-3})} \\ &\sim \underbrace{2(n+2)4^{n-3}}_{\sim 2(n+2)4^{n-3}} \end{aligned}$$

QUESTION:

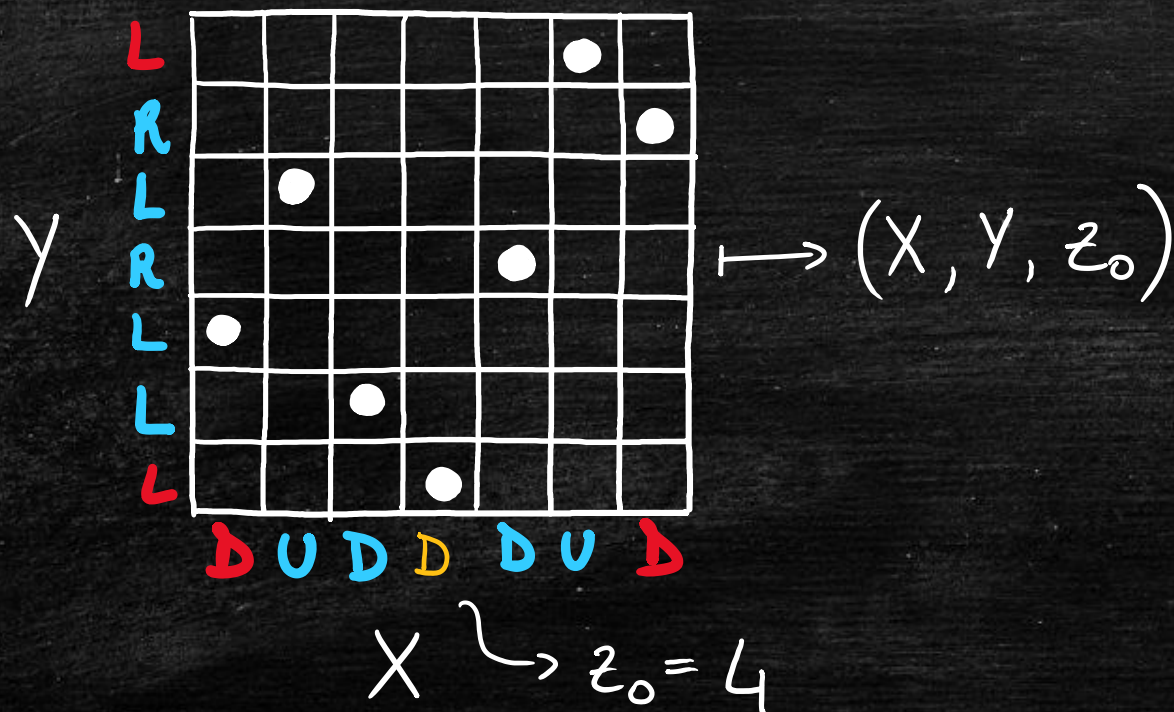
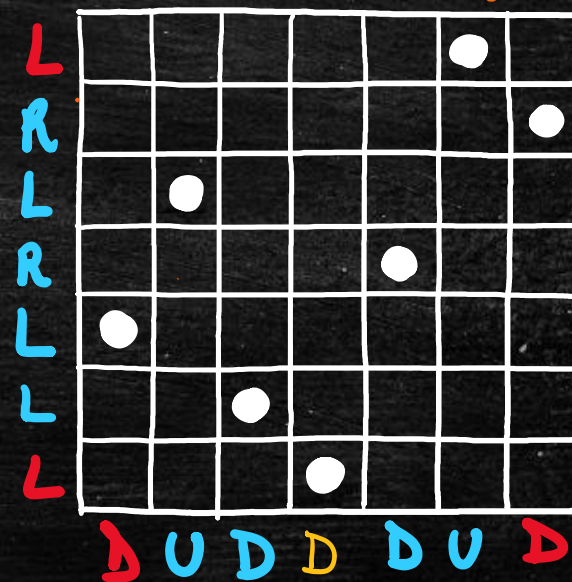
How do we sample a uniform square permutation?

SAMPLING UNIFORM SQUARE PERMUTATIONS

We consider the following projection map:

$$\varphi : Sq(n) \longrightarrow \{U, D\}^n \times \{L, R\}^n \times [n]$$

The space of anchored pairs of sequences of labels.



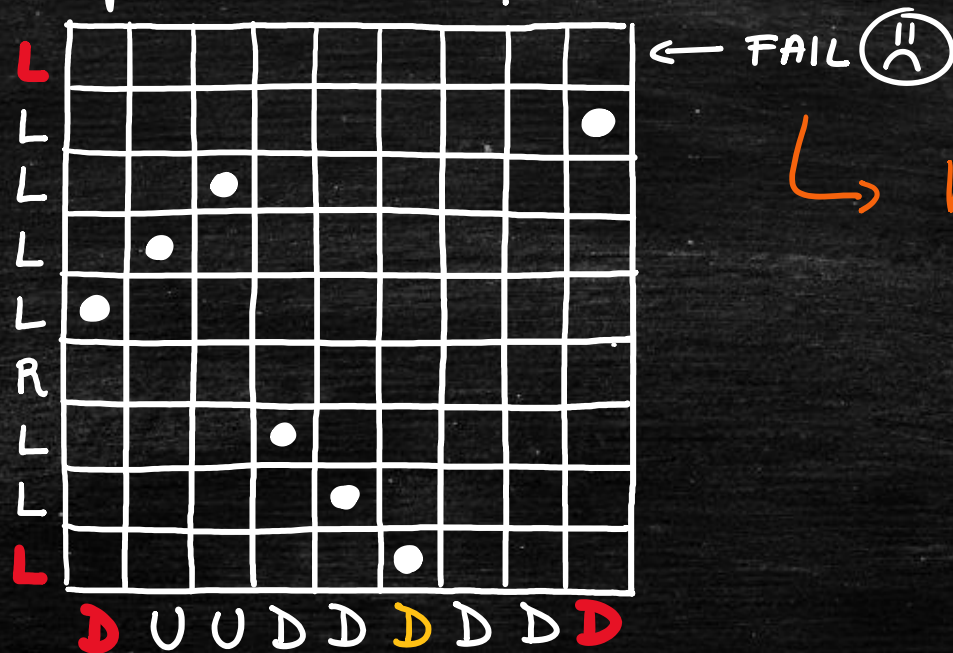
Def: We say that (X, Y, z_0) is a **good** anchored pair of sequences if

$$X_1 = X_n = X_{z_0} = D \quad \text{and} \quad Y_1 = Y_n = L.$$

Note that # of possible good anchored pairs = $2^{n-2}(2 \cdot 2^{n-2} + (n-2)2^{n-3}) = 2(n+2)4^{n-3}$.

PROBLEM: We can not reconstruct a square permutation from all good anchored pairs of sequences.

Counterexample:



↳ Why?

TOO MANY

LS & DS!

We need some regularity conditions on our sequences that are satisfied by "asymptotically almost all" good anchored pairs.

Notation:

- $ct_D(i) = \#$ of D s in X up to (and including) position i
- $pos_D(i) =$ index of the i -th D in X .

PETROV CONDITIONS:

- (1) $|ct_D(i) - ct_D(j) - \frac{1}{2}(i-j)| < n^4$, for $|i-j| < n^6$;
- (2) $|ct_D(i) - ct_D(j) - \frac{1}{2}(i-j)| < \frac{1}{2}|i-j|^6$, for $|i-j| > n^3$;
- (3) $|pos_D(i) - pos_D(j) - 2(i-j)| < n^4$, for $|i-j| < n^6$;
- (4) $|pos_D(i) - pos_D(j) - 2(i-j)| < 2|i-j|^6$, for $|i-j| > n^3$.

Def: We say that a good anchored pair (X, Y, z_0) is regular if:

- X and Y satisfy the Petrov conditions,
- $n^q \leq z_0 \leq n - n^q$

We denote by Ω_n the space of regular anchored pairs of size n .

Lemma 1: $\varphi^{-1} : \Omega_n \longrightarrow \text{Sq}(n)$ is well-defined and injective.

Lemma 2: Let (X, Y, z_0) be chosen independently and uniformly at random from $\{U, D\}^n \times \{L, R\}^n \times [n]$, then $\mathbb{P}((X, Y, z_0) \in \Omega_n) = 1 - o(1)$.

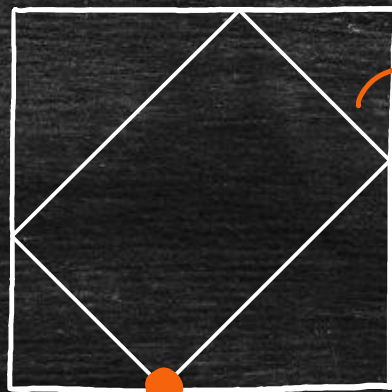
Theorem: With probability $1 - o(1)$ a uniform square permutation σ_n of size n belongs to $\varphi^{-1}(\Omega_n)$.

Proof:
$$\mathbb{P}(\sigma_n \in \varphi^{-1}(\Omega_n)) \underset{\substack{\downarrow \\ \varphi^{-1} \text{ is injective}}}{=} \frac{|\Omega_n|}{|\text{Sq}(n)|} = \frac{\overbrace{2(n+2) 4^{n-3}}^{\text{\# good anchored pairs}} \overbrace{(1 - o(1))}^{\text{Lemma 2}}}{\underbrace{2(n+2) 4^{n-3} (1 - o(1))}_{\text{"enumerative result"}}} \rightarrow 1. \quad \square$$

THEOREM: [SQUARE PERMUTATIONS ARE TYPICALLY RECTANGULAR]

Let σ_n be a uniform random square permutation of size n , then

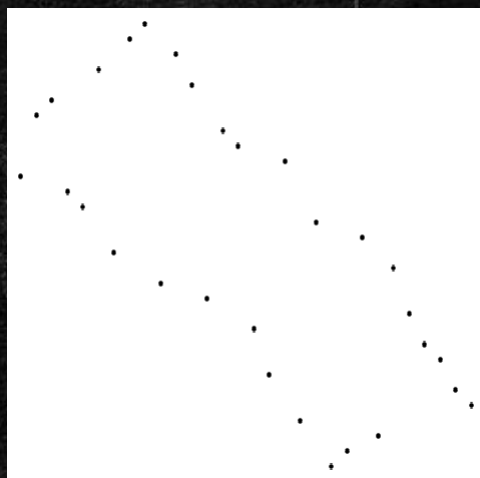
$$\mu_{\sigma_n} \xrightarrow{d} \mu^z =$$



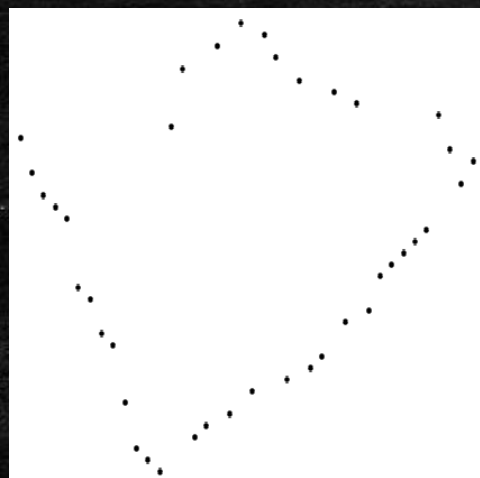
Lebesgue measure on the rectangle with total mass 1.

$$z \sim \text{Unif}([0, 1])$$

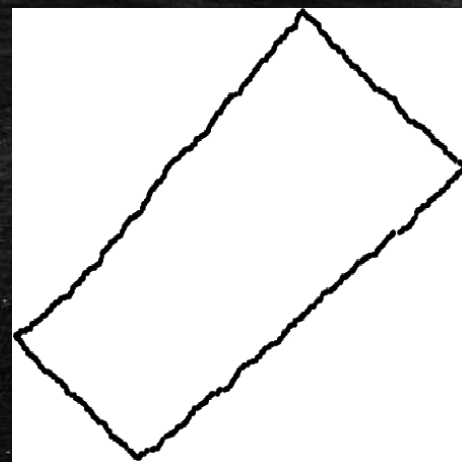
Simulations:



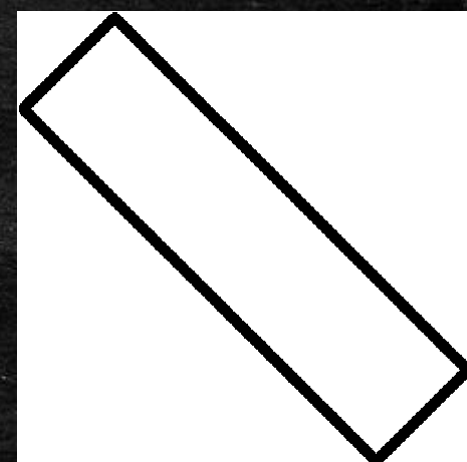
$n=30$



$n=40$

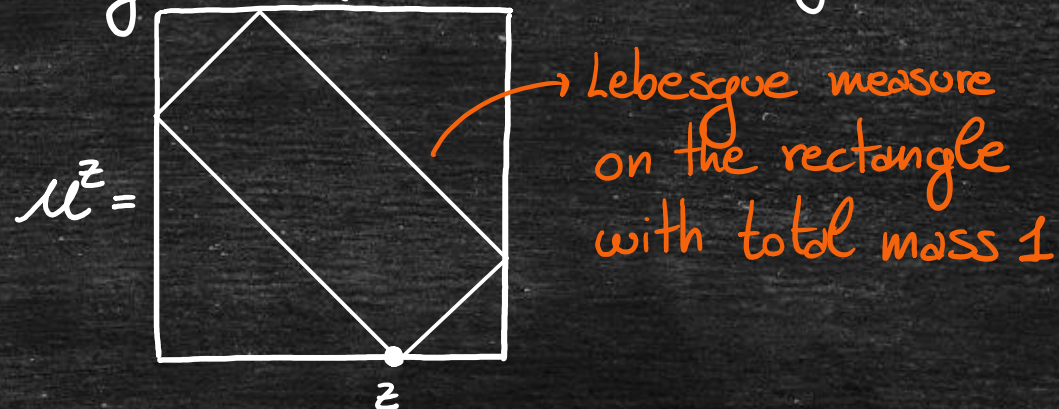


$n=1000$



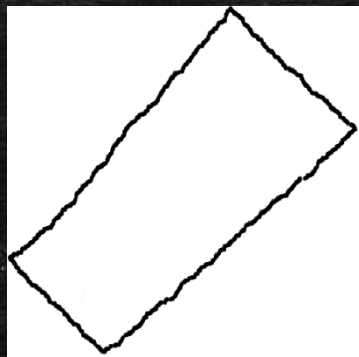
$n=1000000$

Proof: For every $z \in (0,1)$ we define the permuton



Then, for every square permutation $\sigma_n \in \varphi^{-1}(\Omega_n)$, we consider the

permuton $\mu_{\sigma_n} =$



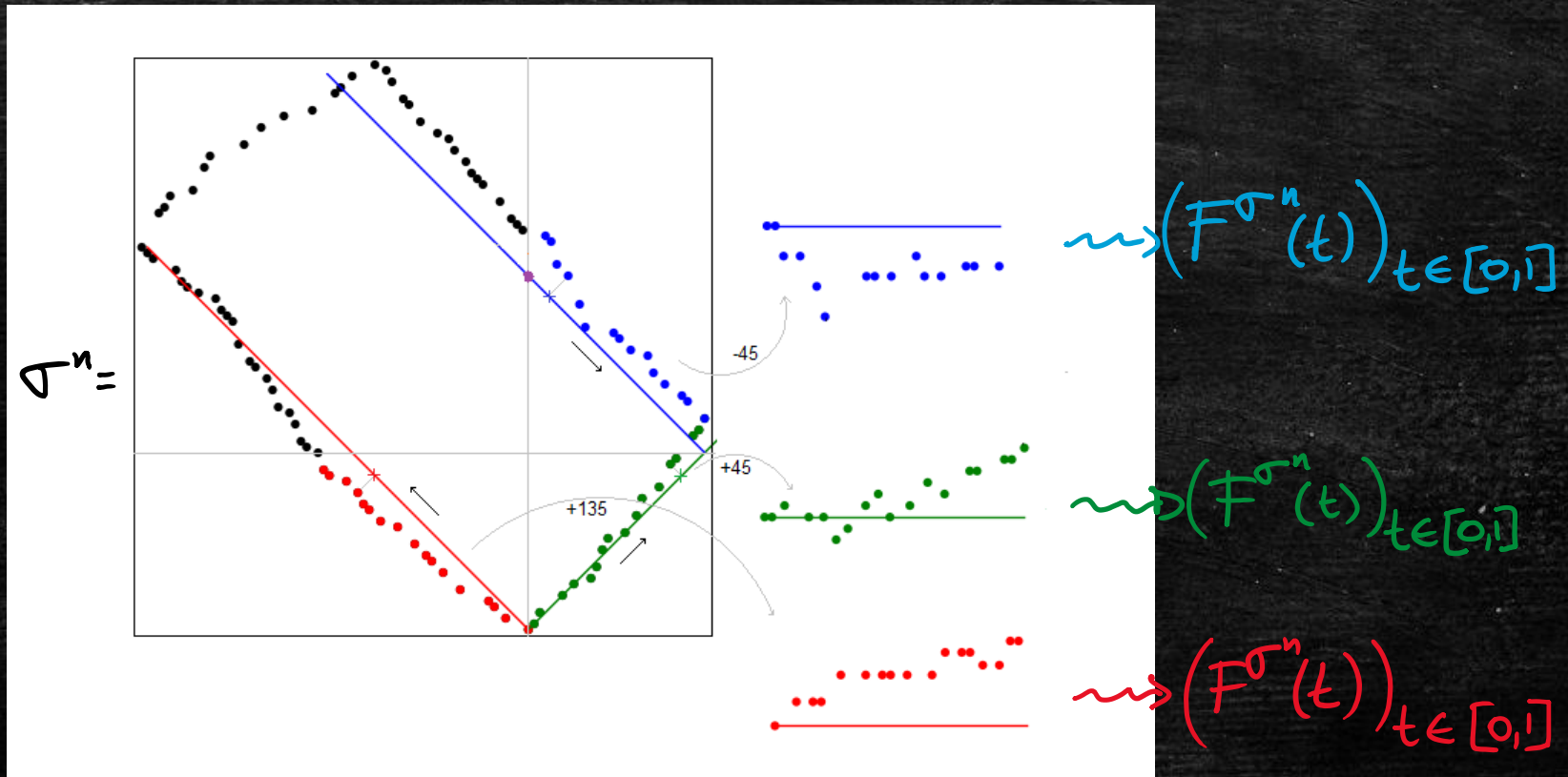
We show that

$$\sup_{\sigma_n \in \varphi^{-1}(\Omega_n)} d_{\square}(\mu_{\sigma_n}, \mu^{z_n}) < C n^{-.4}, \text{ where } z_n = \sigma_n^{-1}(1)/n.$$

$\sigma_n \in \varphi^{-1}(\Omega_n) \hookrightarrow$ metric for the permuton topology.

FLUCTUATIONS

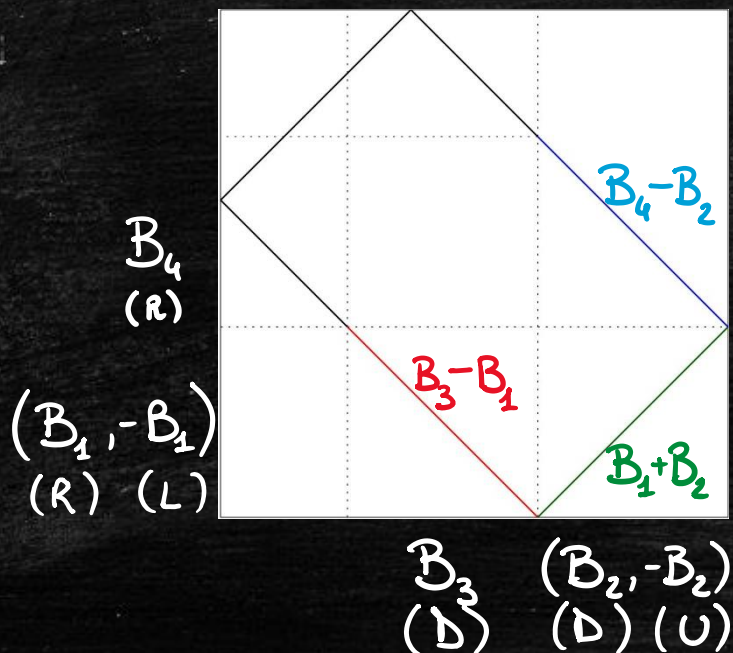
QUESTION: What happens if instead of a factor, n we rescale distances by a factor $\sqrt{n}$?



THEOREM: Let σ^n be a uniform random square permutation of size n , and $B_1(t), B_2(t), B_3(t), B_4(t)$ be four independent standard Brownian motions on the interval $[0,1]$. Conditioning on $z_0 = t_n$, with $\frac{n}{2} + Cn^\epsilon < t_n \leq n - n^\epsilon$, we have the following convergence in distribution

$$\left(F^{\sigma^n}(t), F^{\sigma^n}(t), F^{\sigma^n}(t) \right)_{t \in [0,1]} \xrightarrow{d} \left(B_1(t) + B_2(t), B_3(t) - B_1(t), B_4(t) - B_2(t) \right)_{t \in [0,1]}$$

Proof:



DIFFICULTIES:

- Families of points have random cardinalities
- The interpolating processes are random in both coordinates

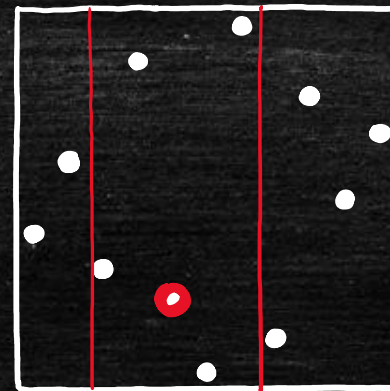
A LOCAL POINT OF VIEW

We can look at square permutations from a local point of view, that is, we look at the neighbourhoods of a random element of a uniformly random square permutations. We study, for all $h \in \mathbb{N}$, the consecutive pattern induced by the h elements on the right and on the left of the chosen element, i.e.

$$r_h(\sigma^n, i_n)$$

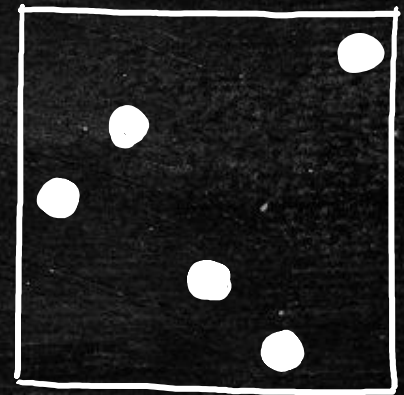
uniform square permutation

uniform index from 1 to n



$$(\sigma^n, 5)$$

r_2



$$r_2(\sigma^n, 5)$$

The presence of two sources of randomness, one for the choice of the permutation and one for the choice of the root, leads to two NON-EQUIVALENT possible analysis:

$$\textcircled{1} \lim_{n \rightarrow \infty} \mathbb{P}(r_n(\sigma^n, i_n) = (\pi, h+1)) \quad \forall \pi \in \mathcal{S}^{2h+1}, h \geq 1 \quad \left[\text{ANNEALED B-S CONVERGENCE} \right]$$

$$\textcircled{2} \lim_{n \rightarrow \infty} \left(\mathbb{P}(r_n(\sigma^n, i_n) = (\pi, h+1) \mid \sigma^n) \right)_{\pi \in \mathcal{S}^{2h+1}, h \geq 1} \quad \left[\text{QUENCHED B-S CONVERGENCE} \right]$$

THEOREM: Let σ^n be a uniform square permutation, then σ^n quenched B-S converges. *for the topology defined in a previous work!*

Remark: We have an explicit construction of the limiting object.

CONNECTION WITH PERMUTATION PATTERNS

Notation:

$$\widetilde{\text{occ}}(\pi, \sigma) = \frac{\# \text{ occurrences of } \pi \text{ in } \sigma}{\binom{|\sigma|}{|\pi|}}$$

$$\widetilde{\text{c-occ}}(\pi, \sigma) = \frac{\# \text{ consecutive occurrences of } \pi \text{ in } \sigma}{|\sigma|}$$

Corollary: Let σ^n be a uniform square permutations, then

$$(\widetilde{\text{occ}}(\pi, \sigma))_{\pi \in \mathcal{S}} \xrightarrow{d} (\Delta_\pi)_{\pi \in \mathcal{S}} \quad \& \quad (\widetilde{\text{c-occ}}(\pi, \sigma))_{\pi \in \mathcal{S}} \xrightarrow{d} (\Delta_\pi)_{\pi \in \mathcal{S}}$$

where $\forall \pi \in \mathcal{S}$, Δ_π (resp. Δ_π) can be described in terms of the permutation (resp. local) limiting object.

WHAT ABOUT "ADDING" INTERNAL POINTS?

(WORK IN PROGRESS WITH DUCHI & SLIVKEN)

Notation: $ASq(n, k)$ = Permutations of size $n+k$ with k internal points

Theorem: Let $k = o(n^{1-\delta})$ for some $\delta > 0$, then

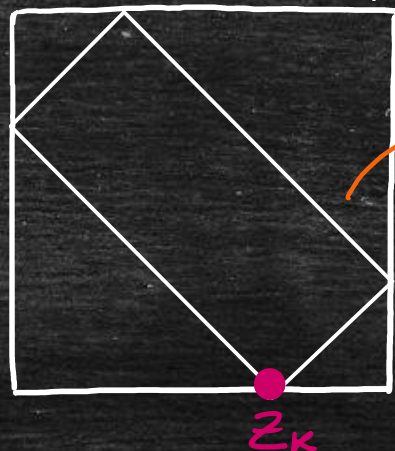
$$|ASq(n, k)| \underset{n \rightarrow \infty}{\sim} \frac{k! 2^{k+1} n^{2k+1} 4^{n-3}}{(2k+1)!}.$$

Q: What happens to the PERMUTON LIMIT?

Theorem: Fix $k > 0$. Let σ_n be a uniform permutation in $Asq(n, k)$,

then

$$\mu_{\sigma_n} \xrightarrow{d} \mu^{z_k} =$$



Lebesgue measure
on the rectangle
with total mass 1

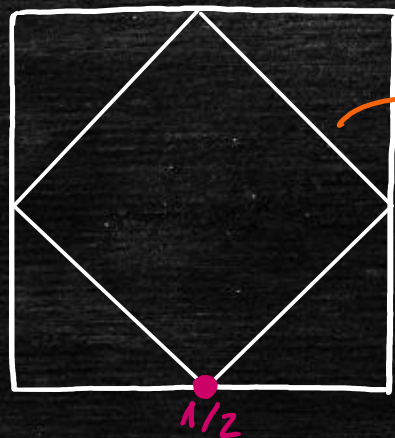
where

$$\mathbb{P}(z_k < s) = (2k+1) \binom{2k}{k} \int_0^s (t(1-t))^k dt \quad \forall s \in (0, 1).$$

Moreover, if $k \rightarrow +\infty$ and $k = o(n^{\frac{1}{2}-\delta})$ for some $\delta > 0$

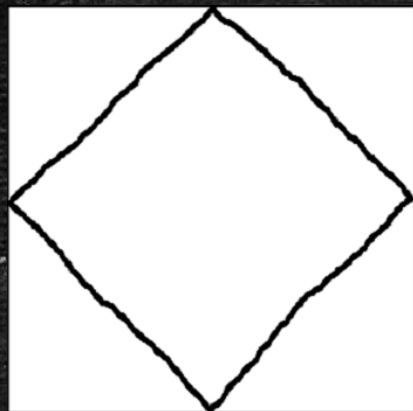
then

$$\mu_{\sigma_n} \xrightarrow{d} \mu^{\frac{1}{2}} =$$

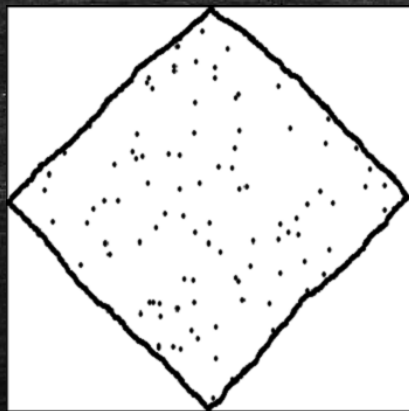


Lebesgue measure
on the square
with total mass 1

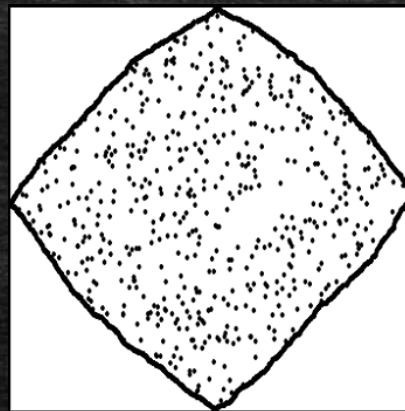
NEXT STEP ?!



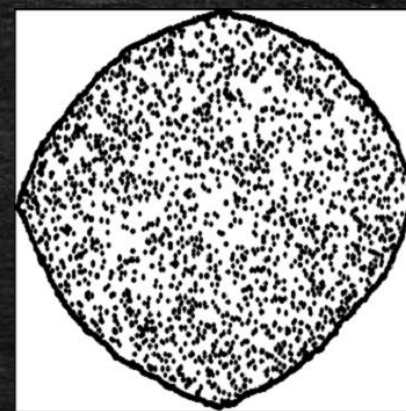
$n = 2000$



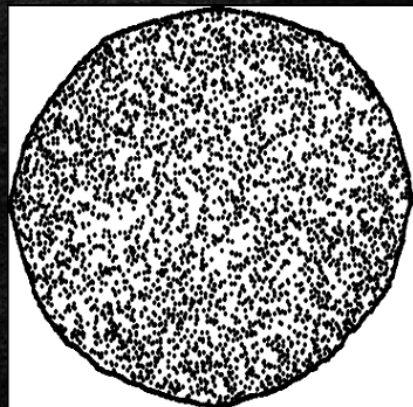
$K = 100$



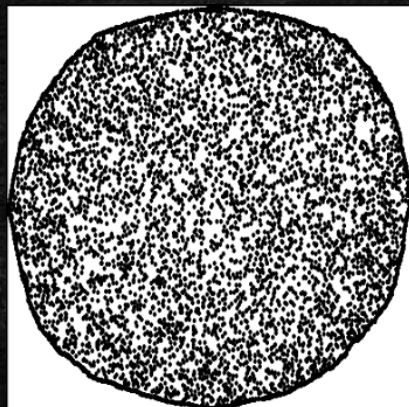
$K = 500$



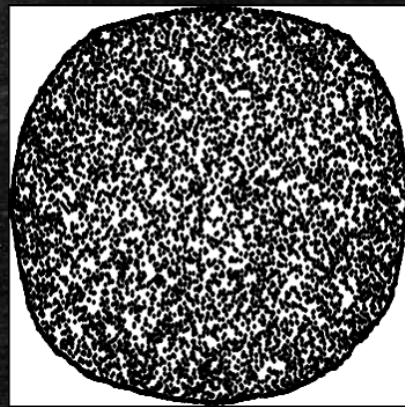
$K = 2000$



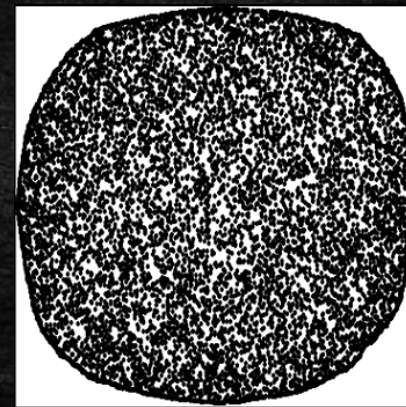
$K = 4000$



$K = 6000$



$K = 8000$



$K = 10'000$

THANK YOU !