



The odd behaviour of the total displacement ratio

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Based on joint work with **Pete Winkler**

Permutation Patterns 2019

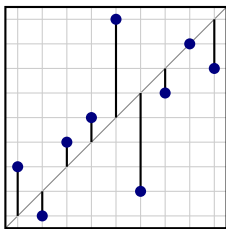
Universität Zürich

21st June 2019

Total displacement

Definition (Knuth)

The *total displacement* of $\sigma \in S_n$ is $\text{td}(\sigma) = \sum_{i=1}^n |\sigma(i) - i|$.



$$\text{td}(314592687) = 2 + 1 + 1 + 1 + 4 + 4 + 1 + 0 + 2 = 16$$

$$\text{inv}(314592687) = 0 + 1 + 0 + 0 + 0 + 4 + 1 + 1 + 2 = 9$$

$\text{td}(\sigma)$ and $\text{inv}(\sigma)$ are both natural measures of how close σ is to the identity.

$$0 \leq \text{td}(\sigma), \text{inv}(\sigma) \leq n^2/2$$

Displacement ratio

Definition

If $\text{inv}(\sigma) > 0$, the *displacement ratio* of σ is $R(\sigma) = \text{td}(\sigma)/\text{inv}(\sigma)$.

Proposition (Diaconis & Graham, 1977)

The displacement ratio satisfies the bounds $1 < R(\sigma) \leq 2$.

Definitions

$\pi_{n,m}$ is a random permutation chosen from $\{\sigma \in S_n : \text{inv}(\sigma) = m\}$.

The *asymptotic displacement ratio* is $R[m(n)] = \lim_{n \rightarrow \infty} \mathbb{E}[R(\pi_{n,m(n)})]$.

Question

How does $R[m(n)]$ behave as $m(n)$ increases from 1 to $\binom{n}{2}$?

Four epochs

$m(n) = n^{1/2}$ (say)	— very sparse (sublinear)
$m(n) = \alpha n$	— sparse (linear)
$m(n) = n^{3/2}$ (say)	— semi-sparse (superlinear subquadratic)
$m(n) = \rho n^2/2$	— dense (quadratic)

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Maximally sparse

- If $\text{inv}(\sigma) = 1$, then $\text{td}(\sigma) = 2$, so $R[1] = 2$.
- If $\text{inv}(\sigma) = 2$, then $\text{td}(\sigma) = 4$, so $R[2] = 2$.

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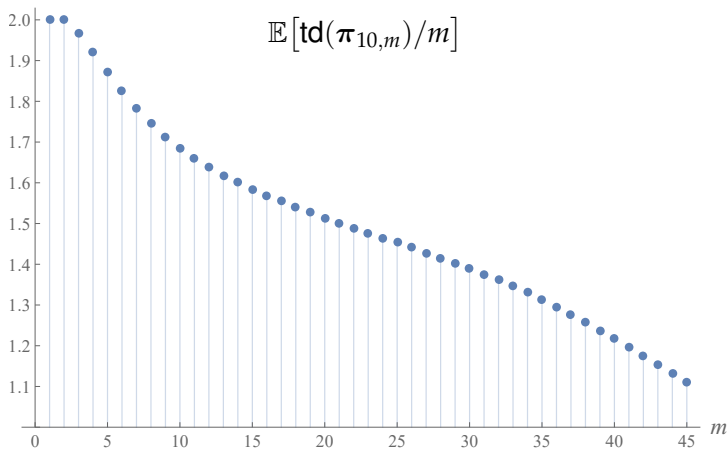
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Maximally dense

$$\text{inv}(n \dots 21) = \binom{n}{2} \qquad \text{td}(n \dots 21) = \lfloor n^2/2 \rfloor$$

$$\lim_{n \rightarrow \infty} R(n \dots 21) = 1$$

Monotonicity?



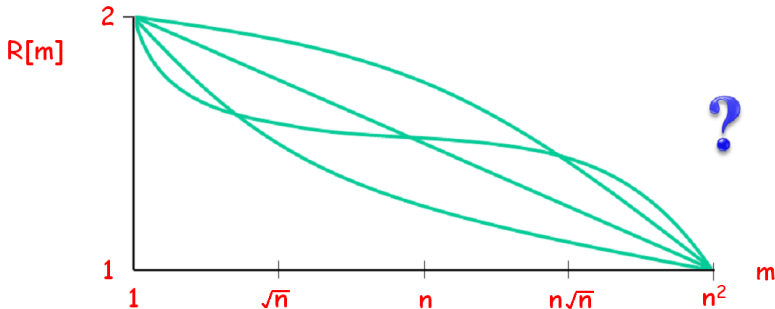
Conjecture

$\mathbb{E}[\text{td}(\pi_{n,m})/m]$ strictly decreases as m increases from 2 to $\binom{n}{2}$.

How does $R[m]$ decrease?

$$R[m(n)] = \lim_{n \rightarrow \infty} \mathbb{E}[\text{td}(\pi_{n,m(n)})/m(n)]$$

How does $R[m]$ behave in the various epochs for $m(n)$?



The dense epoch: $m = \rho n^2/2$

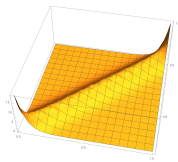
Permutons

- limit object for dense permutations
- probability measure on $[0, 1]^2$ with uniform marginals

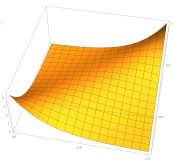
$\lim_{n \rightarrow \infty} \pi_{n, \rho n^2/2} = \mu_\rho$ [Shannon Starr, *J. Math. Phys.*, 2009; KKRW 2015]

$$\mu_\rho(x, y) = \frac{r(e^r - 1)e^{r(x+y-1)}}{(e^{rx} + e^{ry} - e^{r(x+y-1)})^2},$$

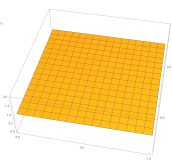
where r is defined by $\rho = \frac{\pi^2 - 6\text{Li}_2(e^r)}{3r^2} + \frac{2 - 2\log(1 - e^r)}{r} + 1$.



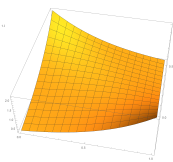
$\rho = 0.1$



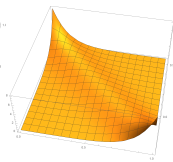
$\rho = 0.3$



$\rho = 0.5$



$\rho = 0.6$



$\rho = 0.8$

The dense epoch: $m = \rho n^2/2$

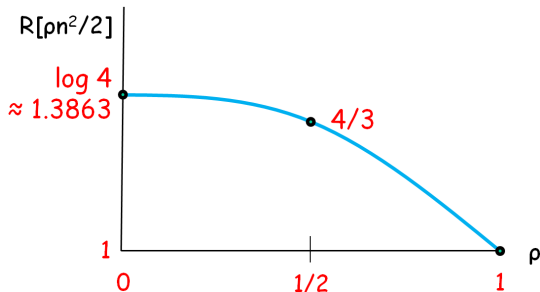
$$R[\rho n^2/2] = \frac{2}{\rho} \int_0^1 \left(\int_0^x (x-y) \mu_\rho(x,y) dy + \int_x^1 (y-x) \mu_\rho(x,y) dy \right) dx$$

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$$\lim_{\rho \rightarrow 0} R[\rho n^2/2] = \log 4$$

$$R[n^2/4] = 4/3$$

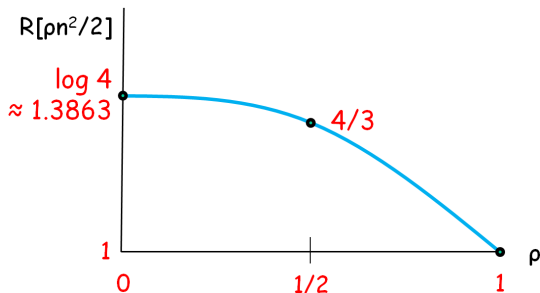


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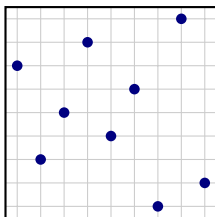
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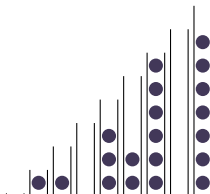
How does $R[m]$ decrease from 2 to $\log 4$ through the sparser epochs?

Inversion sequences

The *inversion sequence* of σ is (I_j) , where $I_j = |\{i : i < j \text{ and } \sigma(i) > \sigma(j)\}|$.



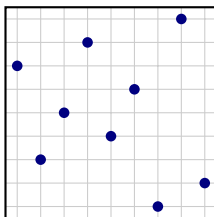
0 1 1 0 3 2 6 0 7



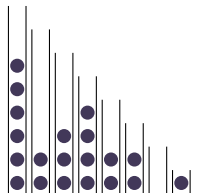
Balls-in-boxes: Sequences (I_j) with $I_j < j$ whose sum is m are in bijection with n -permutations with m inversions: m balls in n boxes.

Dual inversion sequences

Dual inversion sequence of σ : (J_i) , where $J_i = |\{j : i < j \text{ and } \sigma(i) > \sigma(j)\}|$.



6 2 3 4 2 2 0 1 0



Balls-in-boxes: Sequences (J_i) with $J_i \leq n - i$ whose sum is m are in bijection with n -permutations with m inversions: m balls in n boxes.

The sparse epoch: $m = \alpha n$

For large n :

- almost all boxes are effectively unlimited in capacity
- each box is populated independently with X_i many balls, where X_i is geometric with mean α

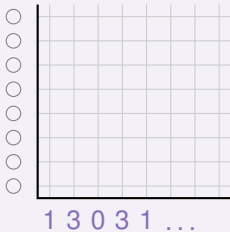


Markov process

- Build a permutation from its dual inversion sequence.
- Determine the stationary distribution.

The Markov process

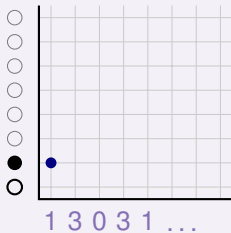
States are binary sequences representing empty and filled *slots*.



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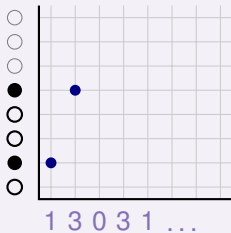
- *From first empty slot to last filled slot.*
- X_i fills the $(X_i + 1)$ th empty slot.



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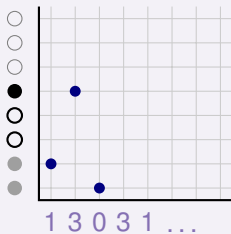
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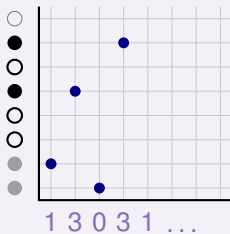
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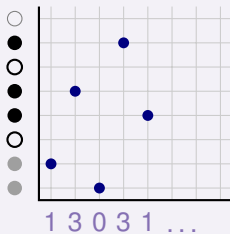
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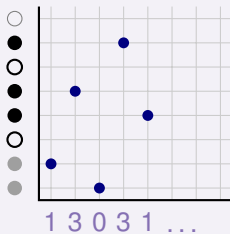
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Determination of the stationary distribution yields ...

The sparse epoch: $m = \alpha n$

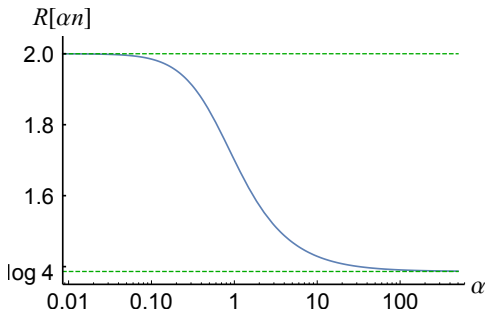
$$R[\alpha n] = 2(1 - q) \sum_{k=1}^{\infty} \frac{(-1)^{k+1} q^{\frac{1}{2}k(k+1)-1}}{1 - q^k}, \quad \text{where } q = \alpha/(\alpha + 1)$$

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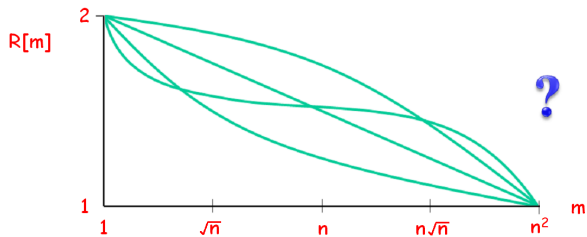
$$\lim_{\alpha \rightarrow 0} R[\alpha n] = 2$$

$$\lim_{\alpha \rightarrow \infty} R[\alpha n] = \log 4$$

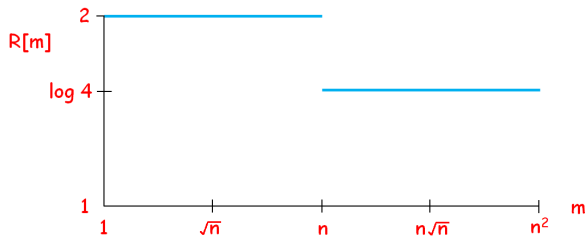


So the linear epoch covers the entire remaining range of R .

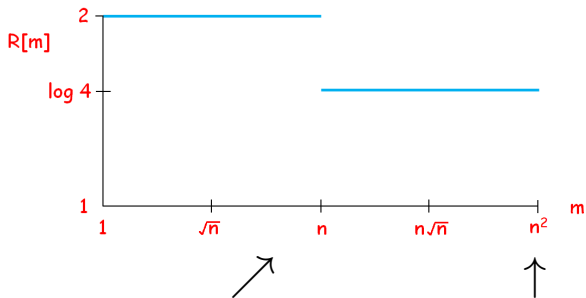
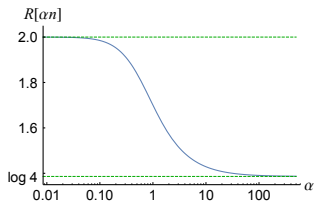
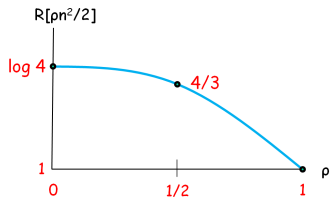
Conclusion



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Final comments

A rough “explanation”?

- m linear: X_i geometric with constant mean.
 - ▶ $(X_i + 1)$ th empty slot is constrained locally.
- m superlinear but subquadratic: mean of X_i grows with n .
 - ▶ Neither locally or globally constrained.
- m quadratic: X_i *truncated* geometric/exponential.
 - ▶ Globally constrained by box capacities.

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Lots still to do

- monotonicity
- concentration
- ...

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Thanks for listening!