

Flip-sort and extremal cases of pop-stack sorting

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Runs and falls

A **run** is a maximal ascending sequence.

A **fall** is a maximal descending sequence.

Each permutation is uniquely split

into runs (by descents)

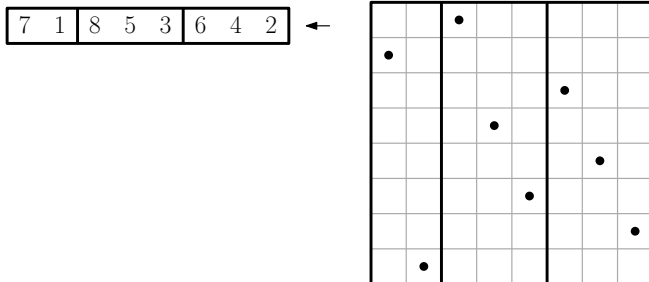
7	1	8	5	3	6	4	2
---	---	---	---	---	---	---	---

into falls (by ascents)

7	1	8	5	3	6	4	2
---	---	---	---	---	---	---	---

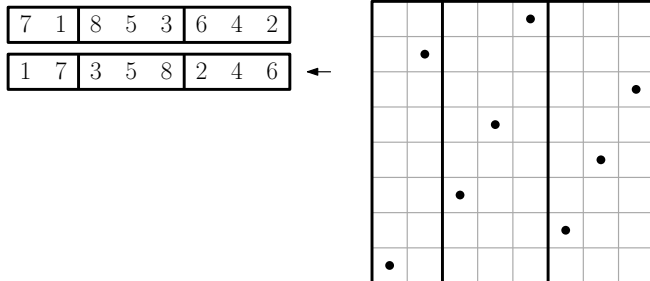
Flip-sort and pop-stack sorting

Flip-sort is a natural sorting algorithm that proceeds by repeated application of **pop-stack sorting**: transformation T that flips all the falls.



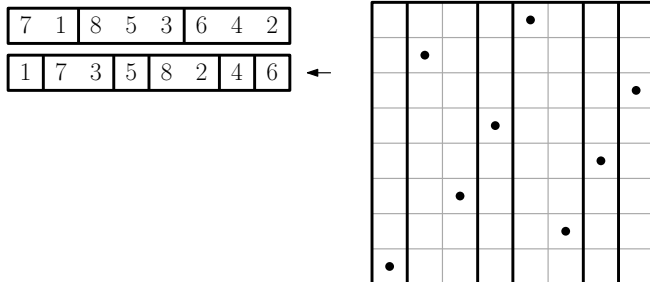
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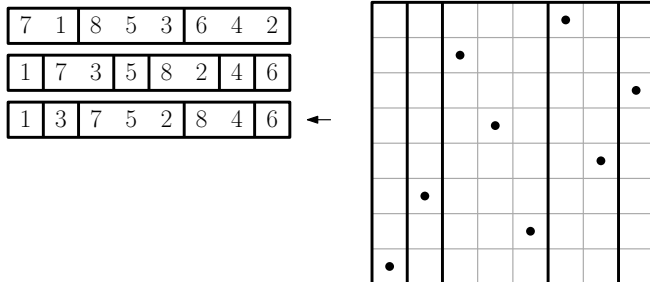
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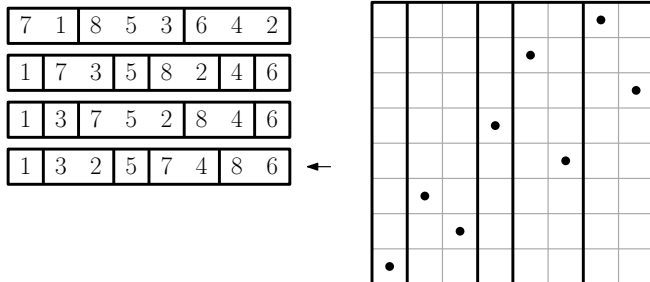
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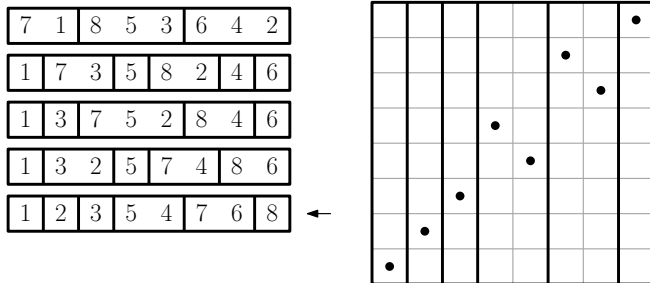
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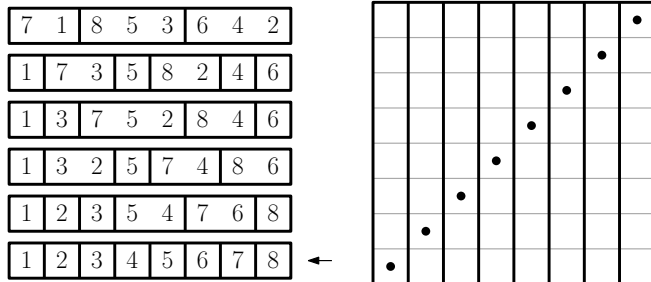
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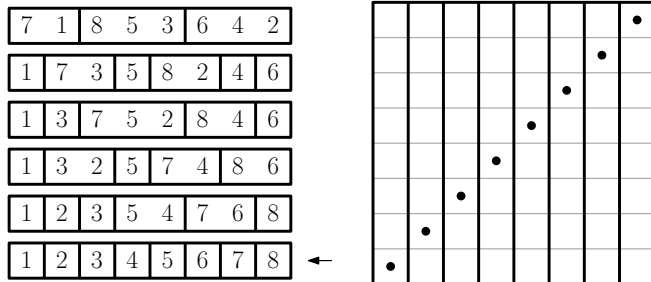
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Theorem [Ungar, 1982]: Each permutation of size n is sorted by at most $n - 1$ iterations of T . Equivalently: $\text{Im}(T^{n-1}) = \{\text{id}\}$.

p -pop-stack-sortability

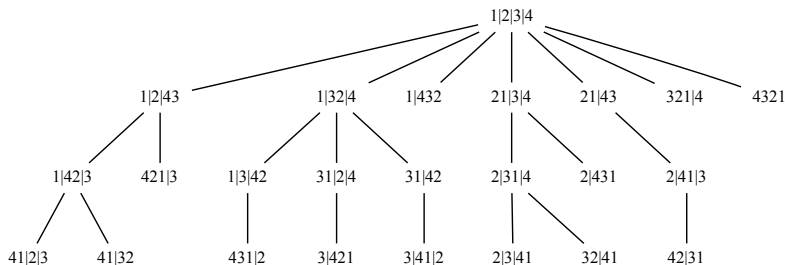
Definition. A permutation π is **p -pop-stack-sortable** if it is sorted by (at most) p iterations of T : $T^p(\pi) = \text{id}$.

[Avis and Newborn, 1981]: A permutation π is 1-pop-stack-sortable ($T(\pi) = \text{id}$) if and only π is layered.

[Pudwell and Smith, 2017]: The generating function for the number of 2-pop-stack-sortable permutations ($T(T(\pi)) = \text{id}$) is rational. A structural characterization of 2-pop-stack-sortable permutations (+representation by forbidden patterns, +bijection with a family of polyominoes).

[Claesson and Guðmundsson, 2017]: For each fixed p , the generating function for the number of p -pop-stack-sortable permutations is rational.

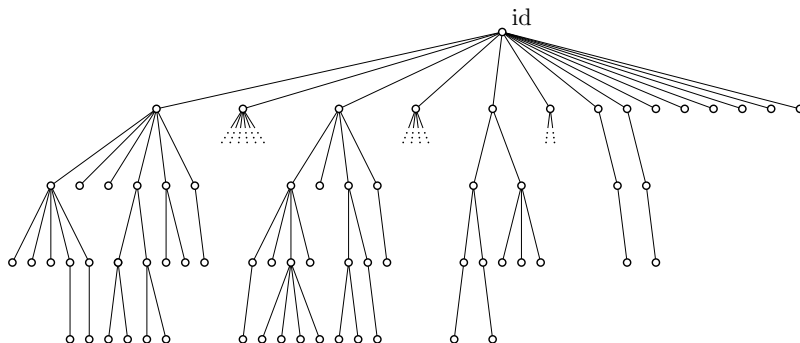
The flip-sort tree for $n = 4$



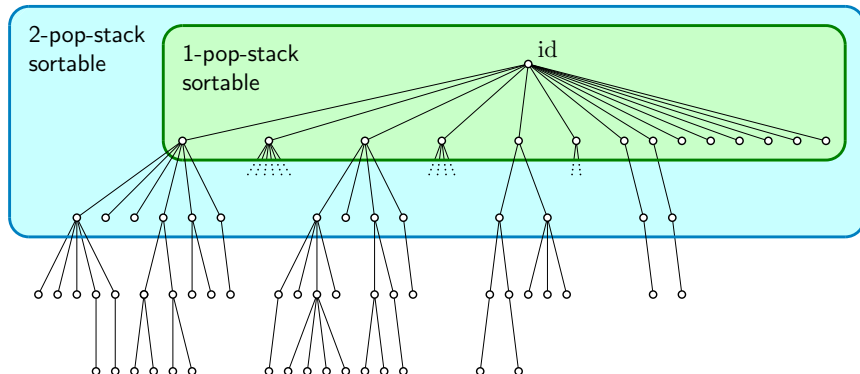
The nodes are the permutations of length $n = 4$.

Each edge (seen as directed upwards) is an application of T , which flips all falls.

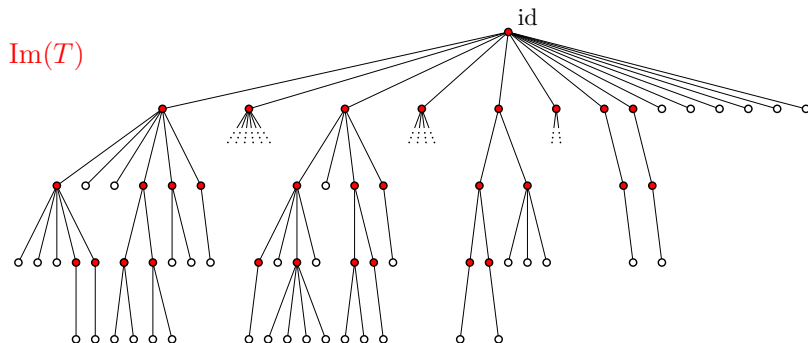
The flip-sort tree for $n = 5$



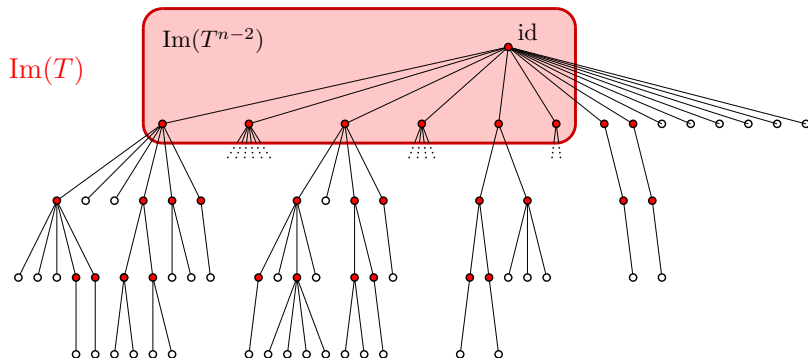
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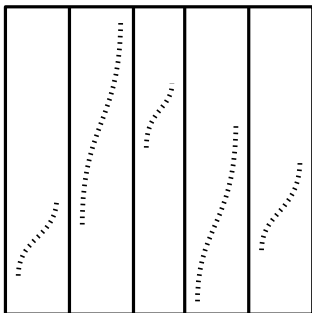
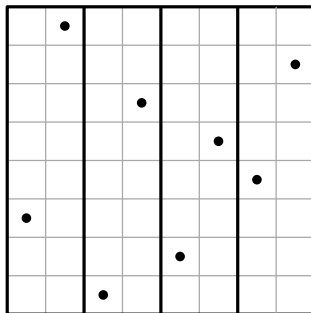
The flip-sort tree for $n = 5$



The image of T

Theorem

$\tau \in \text{Im}(T)$ if and only if the adjacent runs of τ overlap.



$\in \text{Im}(T)$

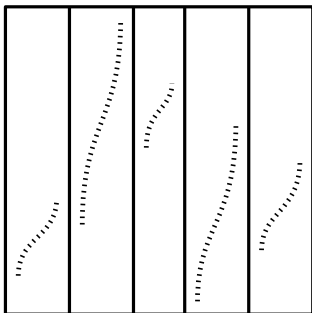
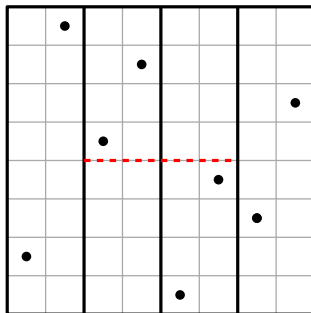
Exact / asymptotic enumeration of $\text{Im}(T)$ is challenging.

See our paper [\[Pop-stack sorting and its image\]](#) (EuroComb 2019).

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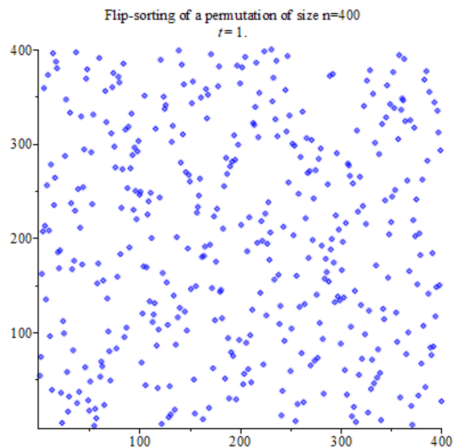


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Flip-sort: the movie

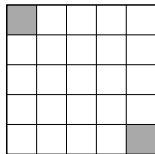


Animation for $n = 400$ (click on the image).

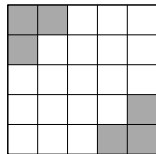
$\text{Im}(T^p)$ and the width of a permutation

Theorem (about the width)

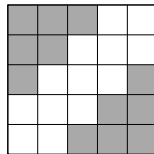
Let $\tau = (a_1 \ a_2 \ \dots \ a_n)$. If $\tau \in \text{Im}(T^p)$, then for each i ($1 \leq i \leq n$) we have $|a_i - i| \leq n - 1 - p$.



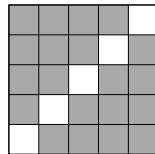
$\text{Im}(T)$



$\text{Im}(T^2)$



$\text{Im}(T^3)$



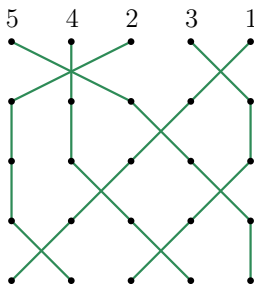
$\text{Im}(T^4)$

(In these images: the theorem says that the grey areas are empty.)

NB: For $p = n - 1$, we recover Ungar's result: $\text{Im}(T^{n-1}) = \{\text{id}\}$.

The wiring diagram

This is the “usual” wiring diagram: paths connect occurrences of the same letter.



5	4	2	3	1
---	---	---	---	---

2	4	5	1	3
---	---	---	---	---

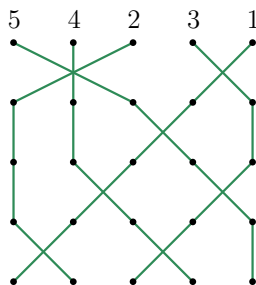
2	4	1	5	3
---	---	---	---	---

2	1	4	3	5
---	---	---	---	---

1	2	3	4	5
---	---	---	---	---

The wiring diagram

This is the “usual” wiring diagram: paths connect occurrences of the same letter.



5	4	2	3	1
2	4	5	1	3
2	4	1	5	3
2	1	4	3	5
1	2	3	4	5

In this proof, another kind of wiring diagrams (introduced by Ungar) is used.

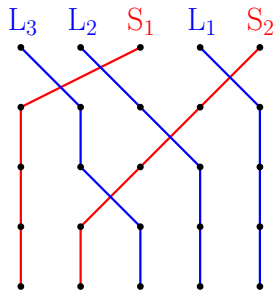
The shadow wiring diagram

Choose k ($1 \leq k \leq n-1$) and denote: $\mathcal{S} = \{1, \dots, k\}$, $\mathcal{L} = \{k+1, \dots, n\}$.
($\mathcal{S} = \text{small}$, $\mathcal{L} = \text{large}$.)

The path S_i connects the i th (left to right) letters from $\mathcal{S}$.

The path L_j connects the j th (right to left) letters from $\mathcal{L}$.

An example with $n = 5$, $k = 2$:



$\mathcal{L} = \{3, 4, 5\}$ $\mathcal{S} = \{1, 2\}$

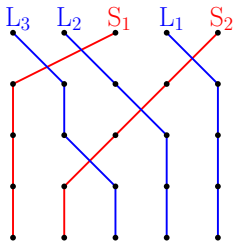
5	4	2	3	1
2	4	5	1	3
2	4	1	5	3
2	1	4	3	5
1	2	3	4	5

The shadow wiring diagram and the dominating paths

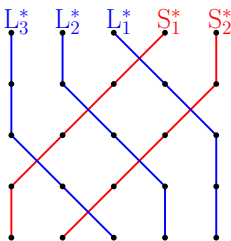
S_i^* and L_j^* are certain **fixed** paths. The crucial fact: S_i is weakly to the left of S_i^* , and L_j^* is weakly to the right of L_j^* . This implies forbidden areas, as claimed.

$$\mathcal{L} = \{3, 4, 5\} \quad \mathcal{S} = \{1, 2\}$$

actual



dominating

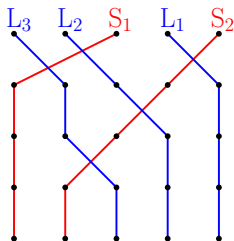


The shadow wiring diagram and the dominating paths

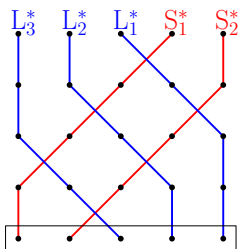
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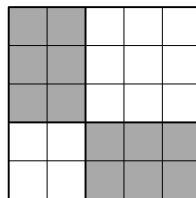
actual



dominating



$k = 2$



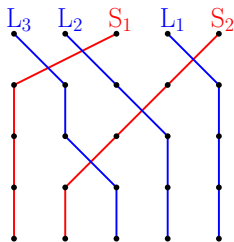
$\text{Im}(T^{n-1})$

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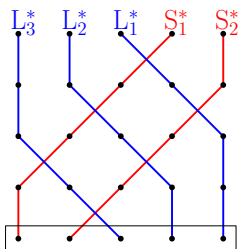
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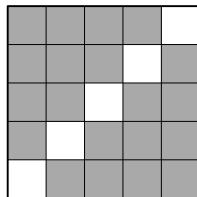
actual



dominating



$k = 1, 2, 3, 4$



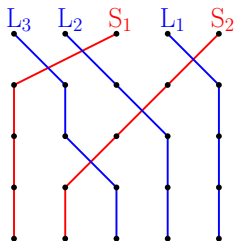
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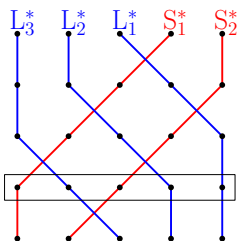
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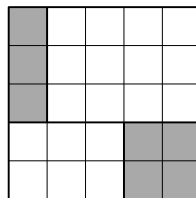
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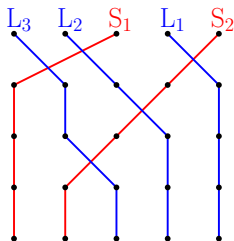
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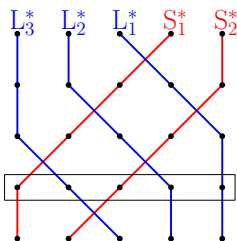
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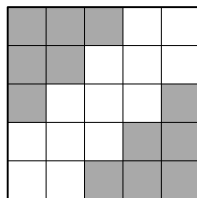
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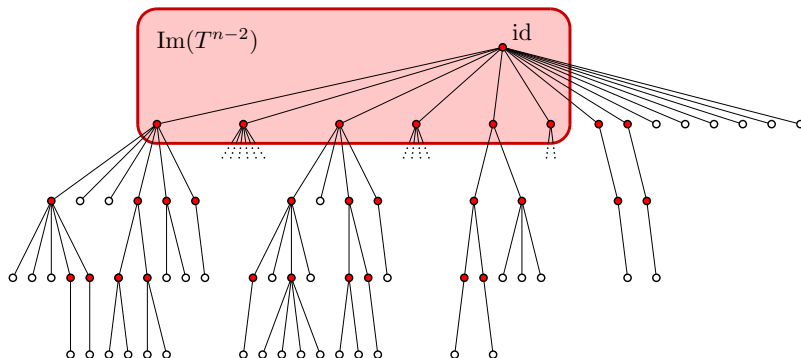


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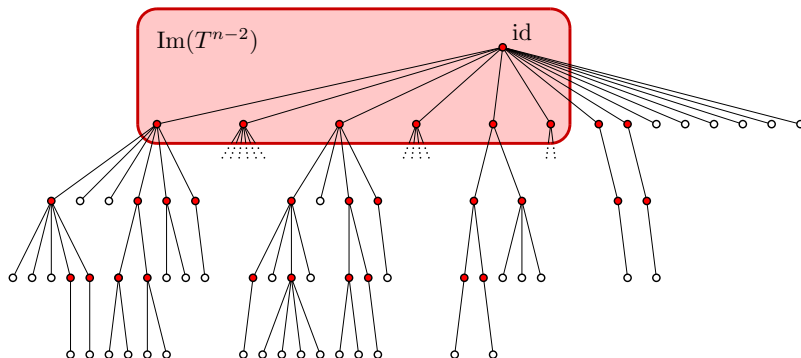


$\text{Im}(T^{n-2})$

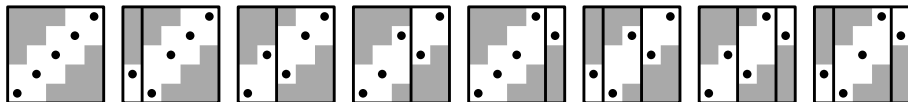
Characterization of $\text{Im}(T^{n-2})$



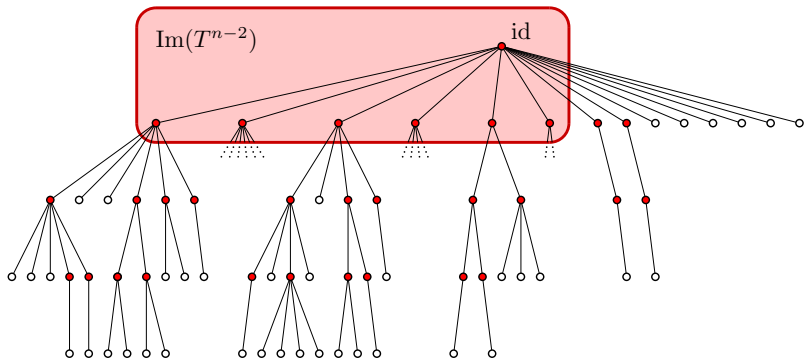
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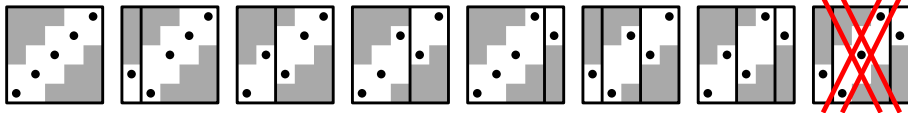
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Characterization of $\text{Im}(\mathcal{T}^{n-2})$

Theorem

A permutation $\tau = a_1 a_2 \dots a_n$ belongs to $\text{Im}(\mathcal{T}^{n-2})$ if and only if

- ① τ is **thin**, that is: $|a_i - i| \leq 1$ for each i , and
- ② τ has no inner runs of odd size.

This implies

$$|\text{Im}(\mathcal{T}^{n-2})| = \begin{cases} 2^{n/2-1} + 2^{n/2} - 1 & \text{for even } n \\ 2^{(n+1)/2} - 1 & \text{for odd } n \end{cases}$$

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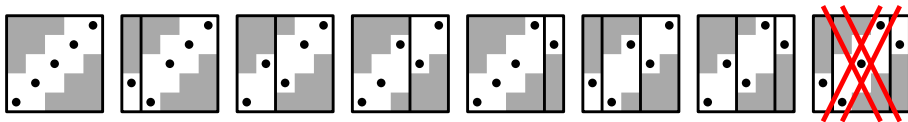
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- 2 τ has no inner runs of odd size.

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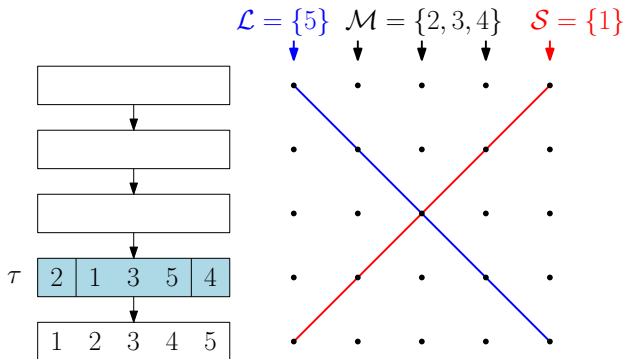
$$|\text{Im}(T^{n-2})| = \begin{cases} 2^{n/2-1} + 2^{n/2} - 1 & \text{for even } n \\ 2^{(n+1)/2} - 1 & \text{for odd } n \end{cases}$$

Proof. (1) If $\tau \in \text{Im}(T^{n-2})$, then it is thin by the theorem about the width.



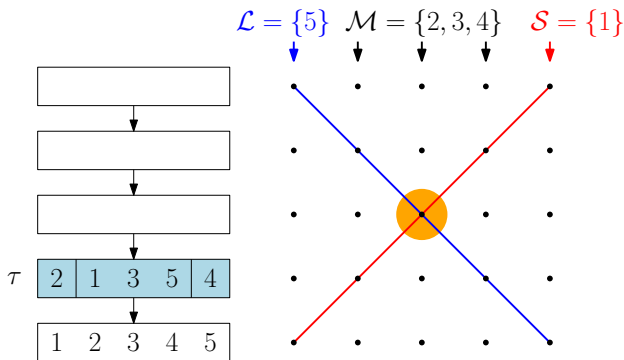
(2) If τ has odd inner runs, then $\tau \notin \text{Im}(T^{n-2})$

Idea of the proof: We adjust the argument with the wiring diagram...



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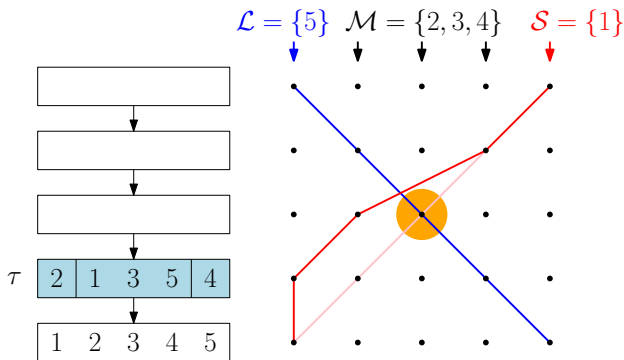
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An odd inner run in $\tau \in \text{Im}(T^{n-2})$ implies a “conflict” between dominating lines.

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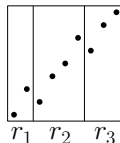
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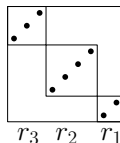
An odd inner run in $\tau \in \text{Im}(T^{n-2})$ implies a “conflict” between dominating lines. As a result, we can modify some of them. It follows that one part ($\mathcal{S}$ or $\mathcal{L}$) is sorted earlier than in $n - 2$ steps — contradiction. (In this example: if the red line is modified, we obtain $\tau = 1 * * * *$.)

(3) If τ is thin without odd inner runs, then $\tau \in \text{Im}(\mathcal{T}^{n-2})$

Let τ be the thin permutation
with runs of sizes $r_1, r_2, \dots, r_s$,
where $r_2, r_3, \dots, r_{s-1}$ are even.



Let π be the skew-layered permutation
with runs of sizes $r_s, r_{s-1}, \dots, r_1$.



Then $\tau = T^{n-2}(\pi)$.

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Then $\tau = T^{n-2}(\pi)$.

Example: $\tau =$

2	1	4	3	5	6	8	7	9
---	---	---	---	---	---	---	---	---

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$$\pi = \begin{array}{|c|c|} \hline 8 & 9 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 4 & 5 & 6 & 7 \\ \hline \end{array} \begin{array}{|c|c|} \hline 2 & 3 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array}$$

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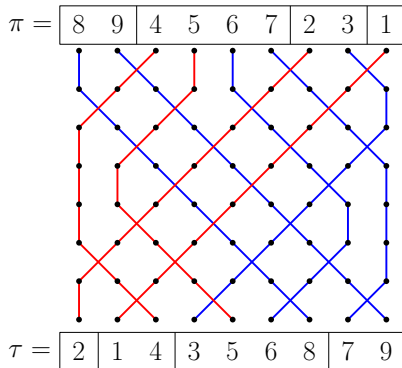
Example: $\tau = \begin{array}{|c|c|c|} \hline 2 & 1 & 4 \\ \hline \end{array} \begin{array}{|c|c|c|c|} \hline 3 & 5 & 6 & 8 \\ \hline \end{array} \begin{array}{|c|c|} \hline 7 & 9 \\ \hline \end{array}$

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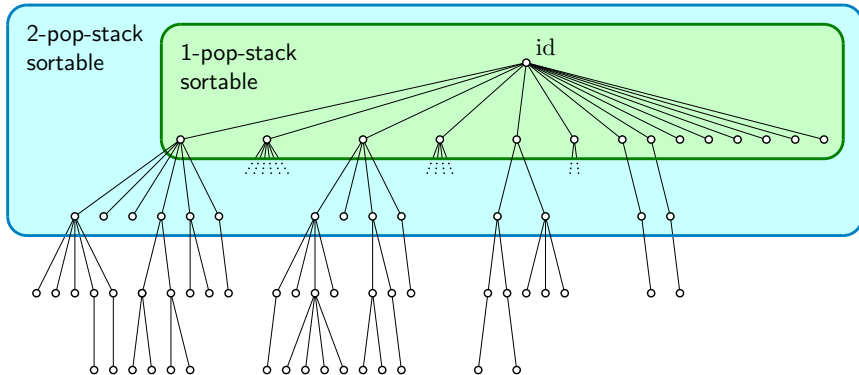
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with runs of sizes $r_s, r_{s-1}, \dots, r_1$.

Then $\tau = T^{n-2}(\pi)$.



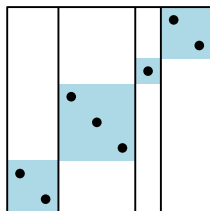
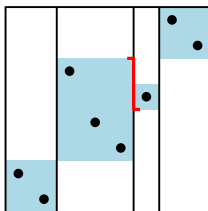
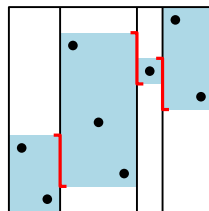
2-pop-stack-sortable permutations



2-pop-stack-sortable permutations: structure

Theorem (Pudwell and Smith, 2017)

A permutation π is 2-pop-stack-sortable, i.e. $T(T(\pi)) = \text{id}$, if and only if adjacent falls of π either do not overlap ($\max(F_i) < \min(F_{i+1})$) or they *have a short overlap* ($\max(F_i) = \min(F_{i+1}) + 1$).



2-pop-stack-sortable permutations: generating function

Theorem (Pudwell and Smith, 2017)

The bivariate generating function for the number of 2-pop-stack-sortable permutations is

$$A(x, y) = \frac{x(1 + x^2y)}{1 - x - xy - x^2y - 2x^3y^2},$$

where x is the variable for the size, and y for the number of ascents.

$y:$	0	1	2	3	4	5	6
$x:$ 1	1						
2	1	1					
3	1	4	1				
4	1	6	8	1			
5	1	8	20	12	1		
6	1	10	36	48	16	1	
7	1	12	56	116	92	20	1

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The sequence $(a_{2n+1, n})$:
1, 4, 20, 116, 708, 4452, ...
[OEIS A085458](#) ???

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↪ bijective proof on next slides!

2-pop-stack-sortable permutations with $\#(\text{asc.})=\#(\text{desc.})$

Proposition

Let $a_{n,k}$ be the number of 2-pop-stack-sortable permutations of size n , with k ascents. The **generating function** of such permutations with $\#(\text{asc.})=\#(\text{desc.})$ is

$$\sum_{n \geq 0} a_{2n+1,n} x^n = \sqrt{\frac{1+x}{1-7x}},$$

and, accordingly, a **closed form** for the coefficients is

$$a_{2n+1,n} = \sum_{i=0}^{n-1} (-1)^i 2^{n-i} \binom{2(n-i)}{n-i} \binom{n-1}{i}.$$

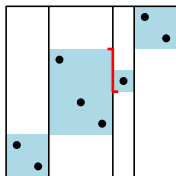
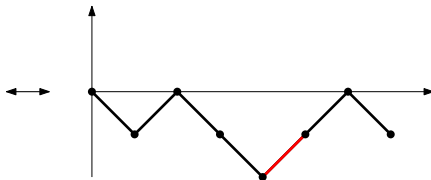
Possible ways of proving:

- 1 Extraction of diagonal coefficients of a bivariate generating function by means of **residue analysis** (+computer algebra on D-finite functions).
- 2 Bijection with **lattice paths**.
(Well developed theory, combinatorial proofs, links to other structures.)

2-pop-stack-sortable permutations and lattice paths

2-pop-stack-sortable permutations are in bijection with Dyck paths such that “U-steps having a D-neighbor are 2-colored”:

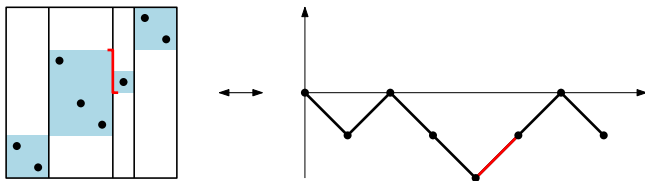
- ascent $\longleftrightarrow U = (1, 1)$ (no fall overlap: black; short fall overlap: red.)
- descent $\longleftrightarrow D = (1, -1)$



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Remark. The condition “red U has at least one neighboring D” can be expressed in terms of **marked patterns in lattice paths**. The **vectorial kernel method** developed in [Asinowski–Bacher–Banderier–Gittenberger, 2019] allows us to enumerate **lattice paths with forbidden/marked patterns**, and, more generally, **combinatorial structures encoded by pushdown automata**.

Proof via lattice paths. Part 1: Dyck paths

The combinatorial specification / generating function for marked Dyck paths:
 x encodes semilength,
 y marks each occurrence of “U-steps with at least one neighboring D-step”.

$$\mathcal{F} = \epsilon \cup (\text{UD} \cup \text{U}(\mathcal{F} \setminus \emptyset)\text{D}) \times \text{Seq}(\text{U}\mathcal{F}\text{D})$$

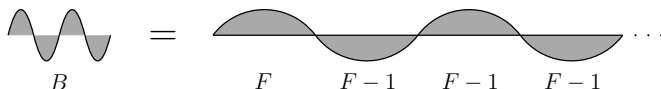
$$\begin{array}{c} \text{Dyck path of semilength } F \end{array} = \begin{array}{c} \bullet \\ 1 \end{array} \cup \begin{array}{c} \text{Dyck path of semilength } xy + x(F-1) \end{array} \begin{array}{c} \text{Dyck path of semilength } xyF \end{array} \begin{array}{c} \text{Dyck path of semilength } xyF \end{array} \dots$$

$$\leadsto F(x, y) = 1 + \frac{xy + x(F(x, y) - 1)}{1 - xyF(x, y)}.$$

$$\leadsto F(x, y) = \frac{1 - x + xy - \sqrt{(1 - x + xy)(1 - x - 3xy)}}{2xy}.$$

Proof via lattice paths. Part 2: Bridges

The coefficients $a_{2n+1,n}$ correspond to **bridges** (walks ending at the x -axis), decomposable as a sequence of marked-Dyck paths $\mathcal{F}$, and rotations of such marked-Dyck paths by 180 degrees (the rotation conserves the pattern constraint, and thus such rotated paths are also counted by $F(x, y)$).

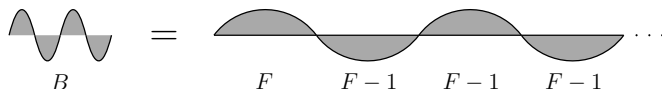


The diagram illustrates the decomposition of a bridge B into a sequence of marked-Dyck paths and their rotations. On the left, a bridge B is shown as a single path starting and ending on the x -axis. On the right, the bridge is decomposed into a sequence of paths: a marked-Dyck path F followed by three rotations of marked-Dyck paths, each labeled $F-1$. The paths are shown as gray-filled areas above and below the x -axis.

$$B(x, y) = \frac{F(x, y)}{1 - (F(x, y) - 1)} = \sqrt{\frac{1 - x + xy}{1 - x - 3xy}}.$$

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So, the generating function of bridges in which the marked U steps are bicolored is

$$B(x, 2) = \sqrt{\frac{1 + x}{1 - 7x}}.$$

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$$B = F \quad F-1 \quad F-1 \quad F-1 \quad \dots$$

$$B(x, y) = \frac{F(x, y)}{1 - (F(x, y) - 1)} = \sqrt{\frac{1 - x + xy}{1 - x - 3xy}}.$$

So, the generating function of bridges in which the marked U steps are bicolored is

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To get a closed formula, we notice that $\sqrt{\frac{1+x}{1-7x}} = \left(\frac{1}{\sqrt{1-4t}} \right) \Big|_{t=\frac{2x}{1+x}}$,

which yields

$$a_{2n+1,n} = \sum_{i=0}^{n-1} (-1)^i 2^{n-i} \binom{2(n-i)}{n-i} \binom{n-1}{i}.$$

Conclusion

Results and interesting links:

- Better understanding of “best case” and “worst case” of flip-sort.
- Links to lattice paths, stacks, automata theory, computational geometry, etc.

Challenging problems:

- Enumeration of $\text{Im}(T)$ [in progress],
- Classification of permutations that satisfy $T^{n-1}(\pi) = \text{id}$ [in progress],
- **Full analysis of flip-sort:** algorithmic cost, typical behavior/shape, etc.

