

Two families of Wilf-equivalences

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Local Wilf-equivalence

Let $\mathcal{C}$ be a permutation class.

Definition

Two permutations $\alpha, \beta \in \mathcal{C}$ are *locally Wilf-equivalent* ($\alpha \sim \beta$) if the classes $\mathcal{C} \cap \text{Av}(\alpha)$ and $\mathcal{C} \cap \text{Av}(\beta)$ have the same enumeration sequence (generating function)

Wilf-collapse

Definition

The class $\mathcal{C}$ exhibits a Wilf-collapse if:

$$|\mathcal{C}_n/\sim| = o(|\mathcal{C}_n|).$$

Last year's talk

Last year's talk

Theorem

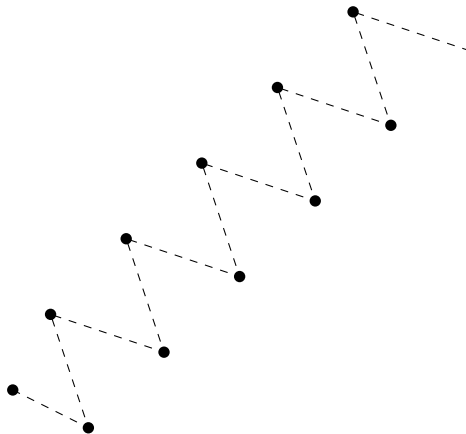
If $\mathcal{C}$ is

- ▶ *sum-closed,*
- ▶ *supercritical, and*
- ▶ *contains two sum-indecomposable permutations whose sum is not contained in a sum-indecomposable permutation,*

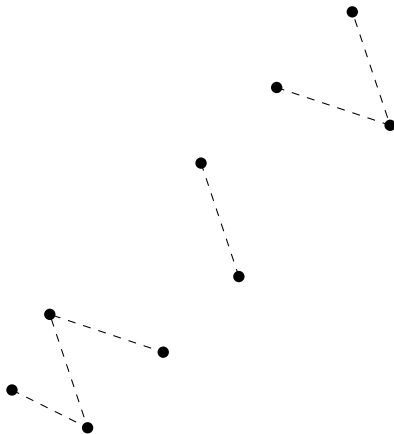
then it exhibits a Wilf-collapse.

If $\mathcal{C}$ contains only finitely many sum-indecomposable permutations then, provided $|\mathcal{C}_n| \rightarrow \infty$, it exhibits a Wilf-collapse.

$\mathcal{SIO}$ (picture)



$\mathcal{SIO}$ (picture)



SIO (words)

Definition

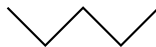
The *Sub*permutations of the *I*ncreasing *O*scillation form a permutation class called SIO . It consists of all permutations contained in the sequence:

$$2, 4, 1, 6, 3, 8, 5, 10, 7, \dots$$

or equivalently all permutations whose inversion graph is a union of paths.

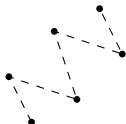
Some sum-indecomposables in $\mathcal{SIO}$

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w_5

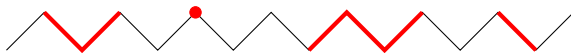
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m_6

There is one of size one (a) one of size two (b) and two of each larger size.

Embedding sums in sum-indecomposables



$$w_3 a m_4 b \leq m_{16}$$

$$w_3 a m_4 b \not\leq m_{14}$$

“Strong” Wilf-equivalence in SIO

Definition

In SIO , $X \equiv Y$ means there is a size and type preserving bijection $\Phi : SIO \rightarrow SIO$ that sends $SIO \cap Av(X)$ to $SIO \cap Av(Y)$.

Type is defined by the slopes of the starting and ending segments of a word each of which can be

up e.g., start of m_6 ,

down e.g., start or finish of w_4 , or

undefined i.e., a or b.

The keys

Lemma

Suppose that $X \equiv Y$ and that P, S are any words. Then $PXS \equiv PYS$.

Lemma

$w_3m_4 \equiv w_4m_3$

Lemma

There is $c > 0$ such that all but an exponentially small proportion of words of size n contain cn factors equal to w_3m_4 .

$\mathcal{X}$

Definition

The class $\mathcal{X} = \text{Av}(2143, 3142, 2413, 3412)$ consists of all permutations that start or finish with their maximum or minimum element, and this is recursively true.

Alternatively, the smallest class closed under the operations:

$$\alpha \mapsto \mathbf{1} \oplus \alpha,$$

$$\alpha \mapsto \mathbf{1} \ominus \alpha,$$

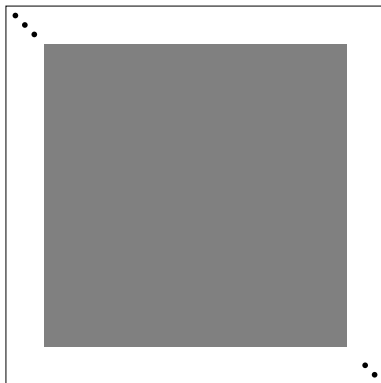
$$\alpha \mapsto \alpha \oplus \mathbf{1},$$

$$\alpha \mapsto \alpha \ominus \mathbf{1}.$$

$\mathcal{X}$ is like an onion

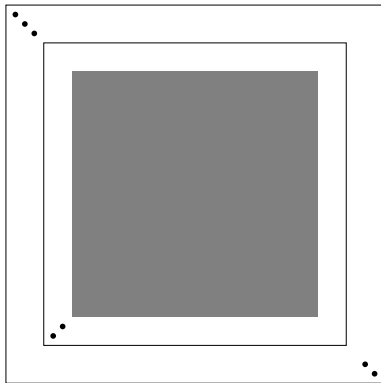


$\mathcal{X}$ is like an onion



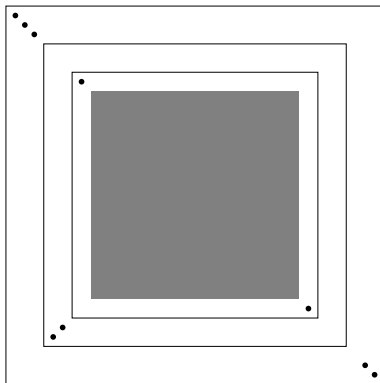
$(-3, -2) \dots$

$\mathcal{X}$ is like an onion



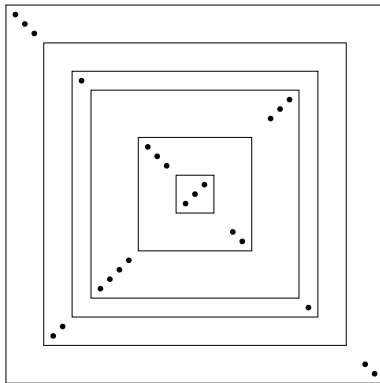
$(-3, -2)(2, 0) \dots$

$\mathcal{X}$ is like an onion



$(-3, -2)(2, 0)(-1, -1) \dots$

$\mathcal{X}$ is like an onion



$(-3, -2)(2, 0)(-1, -1)(4, 3)(-3, -2)(3)$

Recognising involvement in $\mathcal{X}$

To see whether:

$$(a, b)S \cdots \leq t_1 t_2 t_3 \cdots$$

you need to strip away layers until you've seen (a, b) and then check whether S is involved in the remainder.

Recognising involvement in $\mathcal{X}$

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$$(a, b)S \cdots \leq t_1 t_2 t_3 \cdots$$

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Theorem

Any two permutations in $\mathcal{X}$ with the same multiset of layers are Wilf-equivalent (and we can shuffle with sign changes too).

That's a Wilf-collapse from growth rate $2 + \sqrt{2}$ to something sub-exponential.

Thanks:

- ▶ for staying for my talk,
- ▶ for attending PP2019,
- ▶ but especially to Mathilde and all the organisers for a great conference!

See you next year in Indiana!